RESTRICTION OF SAITO-KUROKAWA REPRESENTATIONS

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1. Introduction

This is a write-up of my talk given at the conference on automorphic forms and representation theory of GSp_4, held at Hakuba, Nagano in November 2006. I would like thank Professor Masaaki Furusawa for his invitation to participate in a truly excellent conference, and Professor Hiroshi Saito for supporting my visit with his grant.

In [GP], Gross and Prasad formulated a very precise conjecture describing the branching of an irreducible representation of SO_n when restricted to SO_{n-1} over a local field. Their conjecture, however, assumes the local Langlands correspondence for special orthogonal groups and so can only be checked in cases where one has (at least partially) such a correspondence. This is the case, for example, in many low rank groups, or for certain tamely ramified Langlands parameters. Investigations of the local Gross-Prasad conjecture can be found in a number of papers, such as [P1], [P2] and [GR].

In addition to the local conjecture, there is also a global Gross-Prasad conjecture regarding SO_{n-1}-periods of cusp forms on $SO_n \times SO_{n-1}$. When there is no local obstructions, the non-vanishing of the global period should be controlled by the non-vanishing of a relevant Rankin-Selberg L-function. There have been much significant progress and refinements on this global conjecture recently; see especially [GJR], [BFS] and [II].

The local conjecture of [GP] focuses on addressing the branching problem from SO_n to SO_{n-1} as the representations involved vary over a tempered L-packet; the answer is governed by a condition on epsilon factors. In view of global applications, it is natural to ask how the branching problem would behave if the representations were to vary over a (non-tempered) Arthur packet. The purpose of this talk is to describe the results of my joint paper [GG] with Nadya Gurevich (with an appendix by Gordan Savin [Sa]), in which we investigate this branching problem for one of the best-understood non-tempered Arthur packets, namely the Saito-Kurokawa packets for SO_5.

2. Saito-Kurokawa Representations

Let F be a local field of characteristic zero and fix a non-trivial additive character ψ of F. We begin by recalling the definition and construction of the Saito-Kurokawa A-packets on $PGSp_4$.

The Saito-Kurokawa packets are indexed by irreducible infinite dimensional (unitary) representations of $PGL_2(F)$. Given such a representation π of PGL_2, Waldspurger has associated a packet $\tilde{A}_\pi$ of irreducible genuine (unitary) representations of the metaplectic group $\widetilde{SL}_2(F)$ (see [W1] and [W2]). The local packet $\tilde{A}_\pi$ has two or one element, depending on whether π is a discrete series representation or not. Thus $\tilde{A}_\pi$ has the form

$$\tilde{A}_\pi = \begin{cases} \{\sigma^+, \sigma^-\}, & \text{if } \pi \text{ is a discrete series representation,} \\ \{\sigma^+\} & \text{otherwise.} \end{cases}$$

While the packets themselves are canonical, their parametrization by the representations of PGL_2 depend on the choice of the additive character ψ. With ψ fixed, we shall write

$$\pi = Wd_\psi(\sigma) \quad \text{if } \sigma \in \tilde{A}_\pi.$$

Moreover, if Z is the center of SL_2, its inverse image $\tilde{Z}$ is the center of $\tilde{SL}_2$ and the central character ω_σ of σ has the form

$$\omega_\sigma = \chi_\psi|_{\tilde{Z}} \cdot \epsilon_\psi(\sigma)$$

where χ_ψ is a canonical genuine character (defined in [W1]) of the diagonal torus in $\tilde{SL}_2$ and $\epsilon_\psi(\sigma)$ is a character of Z. We shall regard $\epsilon_\psi(\sigma)$ as ± 1, depending on whether this character is trivial or not. If $\sigma^\epsilon \in \tilde{A}_\pi$, then

$$\epsilon_\psi(\sigma^\epsilon) = \epsilon \cdot \epsilon(\pi, 1/2).$$

Thus the representations in $\tilde{A}_\pi$ can be distinguished by their central characters. Suppose that K is an étale quadratic algebra, corresponding to $a_K \in F^\times/F^{\times 2}$, let ψ_K denote the additive character $\psi_K(x) = \psi(a_K x)$. Then

$$\begin{cases} Wd_{\psi_K}(\sigma) = Wd_\psi(\sigma) \otimes \chi_K \\ \epsilon_{\psi_K}(\sigma) = \epsilon_\psi(\sigma) \cdot \chi_K(-1). \end{cases}$$

The packet $\tilde{A}_\pi$ is constructed by using the local theta lift (associated to ψ) furnished by the dual pairs:

$$PGL_2 \times \tilde{SL}_2 \quad \text{and} \quad PD^\times \times \tilde{SL}_2,$$

where D denotes the unique quaternion division algebra over F. Indeed, we have:

$$\sigma^+ = \theta_\psi(\pi) \quad \text{and} \quad \sigma^- = \theta_\psi(JL(\pi))$$

where $JL(\pi)$ is the Jacquet-Langlands lift of π to $PD^\times$.

Now to construct the Saito-Kurokawa A-packet $SK(\pi)$ of $PGSp_4 \cong SO_5$ associated to π, one considers the theta correspondence furnished by the dual pair

$$\tilde{SL}_2 \times SO_5 \subset \tilde{Sp}_{10}$$

and set

$$\eta^+(\pi) = \theta_\psi(\sigma^+) \quad \text{and} \quad \eta^-(\pi) = \theta_\psi(\sigma^-).$$

Then the Saito-Kurokawa packet (which is independent of ψ) is:

$$SK(\pi) = \{\eta^+(\pi), \eta^-(\pi)\}.$$

The following proposition describes these representations more precisely:

Proposition 2.1. *(i) Let $P = MN$ be the Siegel parabolic of SO_5, with Levi factor $M = PGL_2 \times GL_1$. Let $J_P(\pi, 1/2)$ be the unique irreducible quotient of the normalized induced representation*

$$I_P(\pi, 1/2) = Ind_P^{SO_5} \pi \boxtimes |-|^{1/2}.$$

Then we have

$$\eta^+(\pi) = J_P(\pi, 1/2).$$

(ii) Suppose that $\pi = St$ is the Steinberg representation. Let Q be the other maximal parabolic of SO_5, with Levi factor $L = GL_2$. Then $\eta^-(St)$ is the unique non-generic summand in the normalized induced representation $I_Q(St)$ (which has length 2).

(iii) When π is supercuspidal or a twisted Steinberg representation St_χ (with χ a nontrivial quadratic character), $\eta^-(\pi)$ is supercuspidal.

The above proposition describes the representations in $SK(\pi)$ except when π is supercuspidal or twisted Steinberg, in which case it does not offer any information on $\eta^-(\pi)$. However, there is another way of constructing the packet $SK(\pi)$, using the theta lifting from $GSO(4)$ to GSp_4, and this gives an alternative construction of the supercuspidal representation $\eta^-(\pi)$; see [Sch].

3. **The Main Results**

With the Saito-Kurokawa packets at hand, we can now consider the restriction of $\eta^\epsilon(\pi)$ to the subgroup $SO(2,2) \subset SO(3,2)$. To state our results, we need some facts regarding representations of $SO(2,2)$.

Since $GSO(2,2) \cong (GL_2 \times GL_2)/\Delta\mathbb{G}_m$, one sees that an L-packet on $SO(2,2)$ is indexed by a representation $\tau_1 \boxtimes \tau_2$ of $GSO(2,2)$. The elements of the L-packet are simply the irreducible constituents of the restriction of $\tau_1 \boxtimes \tau_2$ to $SO(2,2)$. We note also that if $\tau_1 \ncong \tau_2^\vee$, then there is a unique irreducible representation of $GO(2,2)$ which contains $\tau_1 \boxtimes \tau_2$ as a constituent; we denote this representation of $GO(2,2)$ by $(\tau_1 \boxtimes \tau_2)^+$. On the other hand, if $\tau_1 \cong \tau_2^\vee$, then the irreducible representation $\tau_1 \boxtimes \tau_2$ of $GSO(2,2)$ has two extensions to $GO(2,2)$. Exactly one of them participates in the similitude theta correspondence between $GO(2,2)$ and GL_2; we denote this extension by $(\tau_1 \boxtimes \tau_2)^+$.

Our main local theorem is:

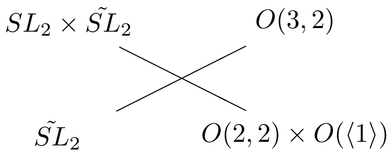
<u>Main Local Theorem</u>

Over a non-archimedean local field of charateristic zero, we have:

(i) $\mathrm{Hom}_{SO(2,2)}(\eta^\epsilon(\pi), \tau_1 \boxtimes \tau_2) = 0$ if $\tau_1 \neq \tau_2^\vee$.

(ii) $Hom_{SO(2,2)}(\eta^\epsilon(\pi), \tau \boxtimes \tau^\vee) \neq 0$ if and only if $\epsilon = \epsilon(\pi \otimes \tau \otimes \tau^\vee, 1/2)$, in which case the dimension of the Hom space is 1.

Let us make a few remarks about the proof of this theorem. Considering the see-saw pair:



one gets the local see-saw identity:

$$\dim \mathrm{Hom}_{SO(2,2)}(\theta_{\psi,0}(\sigma^\epsilon), \tau_1 \boxtimes \tau_2) = \dim \mathrm{Hom}_{\tilde{SL}_2}(\theta_0((\tau_1 \boxtimes \tau_2)^+) \otimes \omega_\psi, \sigma^\epsilon).$$

Here, $\theta_{\psi,0}(\sigma)$ denote the "big" theta lift of σ, i.e. $\sigma \boxtimes \theta_{\psi,0}(\sigma)$ is the maximal σ-isotypic quotient of the Weil representation for $\tilde{SL}_2 x SO(3,2)$. Similarly, $\theta_0((\tau_1 \boxtimes \tau_2)^+$ is the "big" theta lift of $\tau_1 \boxtimes \tau_2)^+$ under the similitude theta correspondence from $GO(2,2) to GL_2$.

Now the theta correspondence from $GO(2,2)$ to GL_2 is well-understood. One knows that $\theta_0(\tau_1 \boxtimes \tau_2)^+) \neq 0$ if and only if $\tau_1 = \tau_2^\vee = \tau$. This already shows (i) of the theorem. To show (ii), one immediately faces the issue that the two "big" theta lifts in the see-saw identity may be reducible. Thus, one needs the following technical lemma:

Lemma 3.1. *(i) For the similitude theta lifting from $GO(2,2)$ to GL_2,*

$$\theta_0((\tau \boxtimes \tau^\vee)^+) = \theta((\tau \boxtimes \tau^\vee)^+) = \tau.$$

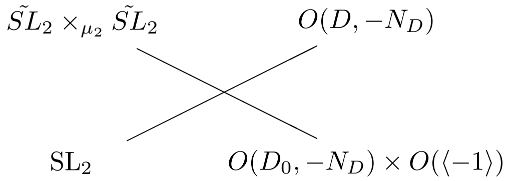
(ii) Consider the theta lift from $\tilde{SL}_2$ to $SO(3,2)$. If σ is not equal to an even Weil representation or the principal series $\tilde{\pi}(|-|^{\pm 3/2})$, then

$$\theta_{\psi,0}(\sigma) = \theta_\psi(\sigma).$$

Thus, we conclude that:

$$\dim \mathrm{Hom}_{SO(2,2)}(\eta^\epsilon(\pi), \tau \boxtimes \tau^\vee) = \dim \mathrm{Hom}_{\tilde{SL}_2}(\tau \otimes \omega_\psi, \sigma^\epsilon) = \dim \mathrm{Hom}_{SL_2}(\tau \otimes \sigma^{\epsilon\vee} \otimes \omega_\psi, \mathbb{C}).$$

and our problem is transferred to that of studying the space of SL_2-invariant trilinear forms on $\tau \otimes \sigma^{\epsilon\vee} \otimes \omega_\psi$. For this, we consider the following see-saw pair:



Here, D is the unique (possibly split) quaternion algebra such that σ^ϵ is the theta lift from $SO(D_0, -N_D)$. By the see-saw identity and the result of Prasad's thesis [P1], we have:

Proposition 3.2. *Let τ be an infinite dimensional representation of GL_2 and σ a representation of $\tilde{SL}_2$. Then for any nontrivial additive character ψ of F,*

$$\mathrm{Hom}_{SL_2}(\tau^\vee \otimes \sigma \otimes \omega_\psi^\vee, \mathbb{C}) \neq 0 \Longleftrightarrow \epsilon(Ad(\tau) \otimes Wd_\psi(\sigma)) = \epsilon_\psi(\sigma),$$

in which case the Hom space is 1-dimensional.

The proposition then implies (ii) of the main local theorem.

Archimedean Case

When F is an archimedean local field, the branching problem has been investigated by Savin [Sa]. It is easy to show that

$$\mathrm{Hom}_{(GL_2 \times GL_2)^0}(\eta^-(\pi), \tau_1 \otimes \tau_2) \neq 0$$

if and only if $\tau_1 = \tau_2^\vee = \tau$ and

$$\epsilon(\pi \otimes \tau \otimes \tau, 1/2) = -1.$$

On the other hand, Savin showed:

$$\mathrm{Hom}_{(GL_2 \times GL_2)^0}(\eta^+(\pi), \tau_1 \otimes \tau_2) \neq 0$$

implies that $\tau_1 = \tau_2^\vee = \tau$ and

$$\epsilon(\pi \otimes \tau \otimes \tau, 1/2) = 1.$$

Moreover, the converse holds if τ is a principal series representation.

Global Case

With these local results at hand, we can now proceed to the global theorem. Suppose that F is a number field with adele ring $\mathbb{A}$ and $\pi = \otimes \pi_v$ is a cuspidal representation of $PGL_2(\mathbb{A})$. There is a global Saito-Kurokawa packet associated to π. A representation in this packet has the form

$$\eta^{\underline{\epsilon}}(\pi) = \otimes_v \eta^{\epsilon_v}(\pi_v).$$

This representation occurs in the space of square-integrable automorphic forms of $PGSp_4$ iff

$$|\underline{\epsilon}| := \prod_v \epsilon_v = \epsilon(\pi, 1/2).$$

We are interested in characterizing the cuspidal representations $\tau_1 \boxtimes \tau_2$ of $SO(2,2) = (GL_2 \times GL_2)^0/\Delta\mathbb{G}_m$ such that the period integral

$$P_{H,\underline{\epsilon}} : (f, \varphi_1, \varphi_2) \mapsto \int_{SO(2,2)(F)\backslash SO(2,2)(\mathbb{A})} f(h) \cdot \varphi_1(h) \cdot \varphi_2(h)\, dh$$

defines a non-zero linear form on $\eta^{\underline{\epsilon}}(\pi) \otimes \tau_1 \otimes \tau_2$. We have:

Main Global Theorem

(i) If the linear form $P_{H,\underline{\epsilon}}$ is non-zero, then $\tau_1 = \tau_2^\vee$.

(ii) Assume that $\tau_1 = \tau_2^\vee = \tau$. There is at most one $\underline{\epsilon}$ for which the linear form $P_{H,\underline{\epsilon}}$ can be nonzero. This distinguished $\underline{\epsilon}$ is characterized by the requirement that

$$\epsilon_v = \epsilon(\pi_v \otimes \tau_v \otimes \tau_v^\vee, 1/2) \quad \text{for all } v.$$

The associated representation occurs in the discrete spectrum iff $\epsilon(\pi \otimes Ad(\tau), 1/2) = 1$.

(iii) Assume that the distinguished representation in (ii) occurs in the discrete spectrum. Then the corresponding linear form $P_{H,\underline{\epsilon}}$ is non-zero if and only if

$$L(\pi \times Ad(\tau), 1/2) \neq 0.$$

We shall not give the proof of the global theorem, but simply remark that it relies ultimately on the recent extension of the Jacquet conjecture (proved by Harris-Kudla [HK]) to the case of an arbitrary étale cubic algebra. This extension is due to Prasad and Schulze-Pilot [PSP].

When the representations involved correspond to holomorphic modular forms of level 1, Ichino has given in [I] a refinement of part (iii) of the theorem by proving an exact formula expressing the value $L(\pi \otimes Ad(\tau), 1/2)$ in terms of the period $P_{H,\underline{\epsilon}}$ evaluated at the modular forms in question. It will be interesting to see if one can obtain such a formula in general, especially in light of the refinement of the global Gross-Prasad conjecture given by Ichino-Ikeda in [II]. One might also ask if the relative trace formula can be brought to bear on such a refinement.

4. **Variants of the Main Results**

In this final section, we obtain some variants of the main local theorem for general forms of (SO_5, SO_4). Thus, we assume once again that F is a p-adic field.

The only inner form of $SO(3,2)$ is the rank one group $SO(4,1)$. In [G], the Saito-Kurokawa packets of $SO(4,1)$ have been analyzed by means of theta lifting from $\tilde{SL}_2$, in analogy with the split case. Fix an infinite-dimensional unitary representation π of PGL_2 with asociated Waldspurger packet $\tilde{A}_\pi = \{\sigma^+, \sigma^-\}$. Then, following [G], set

$$\eta^{+-}(\pi) = \theta_\psi(\sigma^+) \quad \text{and} \quad \eta^{-+}(\pi) = \theta_\psi(\sigma^-).$$

The set $\{\eta^{+-}(\pi), \eta^{-+}(\pi)\}$ is the Saito-Kurokawa packet of $SO(4,1)$ attached to π. Note that it has 2 elements iff π is a discrete series but not the Steinberg representation. Indeed, if $\pi = St$, then $\eta^{+-}(\pi) = 0$ since σ^+ is the odd Weil representation ω_ψ^-.

One may consider the restriction of the Saito-Kurokawa representations from $SO(4,1)$ to the anisotropic inner form $SO(4) = (D^\times \times D^\times)^0/\Delta\mathbb{G}_m$, which is a subgroup of $SO(4,1)$. Here D denotes the quaternion division algebra over F. One has:

Theorem 4.1. *Let τ_1 and τ_2 be discrete series representation of GL_2 and let $JL(\tau_i)$ denote the Jacquet-Langlands lift of τ_i to $D^\times$. Then*

$$\mathrm{Hom}_{SO(4)}(\eta^{\epsilon,-\epsilon}(\pi), JL(\tau_1) \boxtimes JL(\tau_2)) \neq 0 \Longrightarrow \tau_1 = \tau_2^\vee,$$

and

$$\mathrm{Hom}_{SO(4)}(\eta^{\epsilon,-\epsilon}(\pi), JL(\tau) \boxtimes JL(\tau)^\vee) \neq 0 \Longleftrightarrow \epsilon = \epsilon(\pi \otimes \tau \otimes \tau^\vee),$$

in which case the dimension of the Hom space is 1.

In the rest of this section, we consider the restriction of Saito-Kurokawa representations to $SO(3,1)$. The results here are slightly more intricate to state and we begin by introducing some notations for the representations of $SO(3,1)$.

Given any étale quadratic algebra K, there are two quadratic spaces of rank 4 and discriminant K. We denote them by:

$$V_K^+ = \mathbb{H} \oplus (K, N_{K/F}) \quad \text{and} \quad V_K^- = \mathbb{H} \oplus (K, \delta \cdot N_{K/F})$$

where $\mathbb{H}$ denote a hyperbolic plane and $\delta \in F^\times \smallsetminus N_{K/F}(K^\times)$. The associated orthogonal groups are isomorphic. In particular, we have:

$$GSO(V_K^\epsilon) \cong GL_2(K) \times F^\times/\Delta K^\times,$$

with $K^\times$ embeded diagonally via:

$$a \mapsto (diag(a,a), N_{K/F}(a)^{-1}).$$

A representation of $GSO(V_K^\epsilon)$ is thus of the form $\Sigma \boxtimes \chi$, where Σ is an irreducible representation of $GL_2(K)$ whose central character ω_Σ satisfies

$$\omega_\Sigma = \chi \circ N_{K/F}.$$

The subgroup $SO(V_K^\epsilon)$ is isomorphic to $GL_2(K)^0/F^\times$, where

$$GL_2(K)^0 = \{g \in GL_2(K) : \det(g) \in F^\times\}.$$

The embedding $GL_2(K)^0/F^\times \hookrightarrow GSO(V_K^\epsilon)$ is given by:

$$g \mapsto (g, \det(g)^{-1}).$$

An L-packet of $SO(V_K^\epsilon)$ is thus given by the constituents of the restriction of a representation of $GSO(V_K^\epsilon)$ (or equivalently, the restriction of a representation of $GL_2(K)/F^\times$).

We have an embedding of quadratic spaces

$$V_K^+ \hookrightarrow \mathbb{H}^2 \oplus \langle 1 \rangle$$

and thus an embedding

$$SO(V_K^+) \hookrightarrow SO(3,2).$$

On the other hand, V_K^- does not embed into $\mathbb{H}^2 \oplus \langle 1 \rangle$. Rather,

$$V_K^- \hookrightarrow \mathbb{H} \oplus (D_0, -N_D)$$

and so we have

$$SO(V_K^-) \hookrightarrow SO(4,1).$$

One may consider the theta correspondence for the similitude dual pair $GL_2^+ \times GO(V_K^\epsilon)$, which has been studied in [Co] and [R]. Recall that if τ is an irreducible infinite-dimensional representation of GL_2, then the restriction of τ to GL_2^+ is reducible iff $\tau \otimes \chi_K \cong \tau$, in which case there are two constituents. We may label the two constituents by τ^+ and τ^-, so that τ^ϵ occurs in the theta correspondence with $GO(V_K^\epsilon)$ but not with $GO(V^{-\epsilon})$. On the other hand, if τ is irredcuible when restricted to GL_2^+, then τ occurs in the theta correspondence with both $GO(V_K^\epsilon)$ and we simply set $\tau^+ = \tau^- = \tau$. The following lemma describes the theta correspondence for GL_2^+ and $GO(V_K^\epsilon)$:

Lemma 4.2. *(i) Let τ be an irreducible infinite-dimensional unitary representation of GL_2. Then as a representation of $GSO(V_K^\epsilon)$,*

$$\theta_0(\tau^\epsilon) = \theta(\tau^\epsilon) = \Sigma_\tau := BC_K(\tau) \otimes (\omega_\tau \cdot \chi_K),$$

where $BC_K(\tau)$ is the base change of τ to $GL_2(K)$ and ω_τ is the central character of τ.

(ii) Let Σ be an infinite-dimensional unitary representation of $GO(V_K^\epsilon)$, then

$$\theta_0(\Sigma) \neq 0 \Longrightarrow \Sigma|_{GSO(V_K^\epsilon)} = \Sigma_\tau.$$

Moreover, of the two possible extensions of Σ_τ to $GO(V_K^\epsilon)$, exactly one of them, denoted by $\Sigma_\tau^\dagger$, participates in the theta correspondence and one has:

$$\theta_0(\Sigma_\tau^\dagger) = \theta(\Sigma_\tau^\dagger) = \tau^\epsilon.$$

Now one has:

Theorem 4.3. *Consider the restriction of $\eta^\epsilon(\pi)$ from $SO(3,2)$ to $SO(V_K^+)$.*

(i) For an infinite dimensional unitary representation Σ of $GSO(V_K^+) = (GL_2(K) \times F^\times)/\Delta K^\times$, we have:

$$\mathrm{Hom}_{SO(V_K^+)}(\eta^\epsilon(\pi), \Sigma) \neq 0 \Longrightarrow \Sigma = \Sigma_\tau$$

for some infinite dimensional unitary representation τ of $GL_2(F)$.

(ii) If $\tau \otimes \chi_K \neq \tau$, then

$$\mathrm{Hom}_{SO(V_K^+)}(\eta^\epsilon(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \epsilon_{\psi_K}(\sigma^\epsilon) = \epsilon(Ad(\tau) \otimes (\pi \otimes \chi_K))$$

or equivalently

$$\epsilon = \epsilon((\pi \otimes \chi_K) \otimes Ad(\tau)) \cdot \chi_K(-1) \cdot \epsilon(\pi),$$

in which case the Hom space has dimension 1.

(iii) If $\tau \otimes \chi_K = \tau$, then

$$\mathrm{Hom}_{SO(V_K^+)}(\eta^-(\pi), \Sigma_\tau) = 0$$

whereas

$$\mathrm{Hom}_{SO(V_K^+)}(\eta^+(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \epsilon((\pi \otimes \chi_K) \otimes Ad(\tau)) \cdot \chi_K(-1) \cdot \epsilon(\pi) = 1,$$

in which case the Hom space has dimension 1.

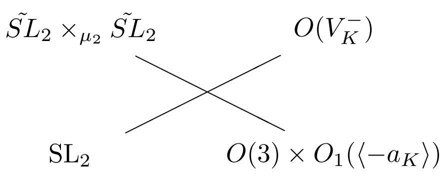
Proof. We give a sketch of the proof, so as to illustrate why the extra complexity in (iii) occurs. Suppose that K corresponds to $a_K \in F^\times/F^{\times 2}$. By using the see-saw

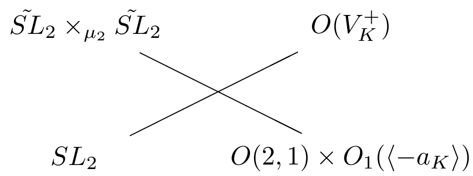
$$\begin{array}{ccc} SL_2 \times \tilde{SL}_2 & & O(3,2) \\ & \times & \\ \tilde{SL}_2 & & O(V_K^+) \times O_1(\langle a_K \rangle) \end{array}$$

one deduces (i) immediately. Moreover, if $\Sigma = \Sigma_\tau$, then

$$\mathrm{Hom}_{SO(V_K^+)}(\eta^\epsilon(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \mathrm{Hom}_{SL_2}(\tau^{+\vee} \otimes \sigma^\epsilon \otimes \omega_{\psi_K}^\vee, \mathbb{C}) \neq 0.$$

If $\tau \otimes \chi_K \neq \tau$, then $\tau^+ = \tau$ and so (ii) follows from Prop. 3.2. Finally, if $\tau \otimes \chi_K = \tau$, then one cannot use Prop. 3.2 directly. Instead, consider the two companion see-saws





Since the theta lift of τ^+ to $GO(V_K^-)$ is zero, the first see-saw gives

$$\mathrm{Hom}_{SL_2}(\tau^{+\vee} \otimes \sigma^- \otimes \omega_{\psi_K}^\vee, \mathbb{C}) = 0$$

which implies the vanishing result of (iii). Similarly, the second see-saw allows one to conclude that

$$\mathrm{Hom}_{SL_2}(\tau^{-\vee} \otimes \sigma^+ \otimes \omega_{\psi_K}^\vee, \mathbb{C}) = 0,$$

so that

$$\mathrm{Hom}_{SL_2}(\tau^\vee \otimes \sigma^+ \otimes \omega_{\psi_K}^\vee, \mathbb{C}) = \mathrm{Hom}_{SL_2}(\tau^{+\vee} \otimes \sigma^+ \otimes \omega_{\psi_K}^\vee, \mathbb{C}).$$

Together with Prop. 3.2, this implies the second part of (iii). □

Similarly, one has:

Theorem 4.4. *Consider the restriction of $\eta^{\epsilon,-\epsilon}(\pi)$ from $SO(4,1)$ to $SO(V_K^-)$.*

(i) For an infinite dimensional unitary representation Σ of $GSO(V_K^-) = (GL_2(K) \times F^\times)/\Delta K^\times$, we have:

$$\mathrm{Hom}_{SO(V_K^-)}(\eta^{\epsilon,-\epsilon}(\pi), \Sigma) \neq 0 \Longrightarrow \Sigma = \Sigma_\tau$$

for some infinite dimensional unitary representation τ of $GL_2(F)$.

(ii) If $\tau \otimes \chi_K \neq \tau$, then

$$\mathrm{Hom}_{SO(V_K^-)}(\eta^{\epsilon,-\epsilon}(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \epsilon_{\psi_K}(\sigma^\epsilon) = \epsilon(Ad(\tau) \otimes (\pi \otimes \chi_K))$$

or equivalently

$$\epsilon = \epsilon((\pi \otimes \chi_K) \otimes Ad(\tau)) \cdot \chi_K(-1) \cdot \epsilon(\pi),$$

in which case the Hom space has dimension 1.

(iii) If $\tau \otimes \chi_K = \tau$, then

$$\mathrm{Hom}_{SO(V_K^-)}(\eta^{+-}(\pi), \Sigma_\tau) = 0$$

whereas

$$\mathrm{Hom}_{SO(V_K^-)}(\eta^{-+}(\pi), \Sigma_\tau) \neq 0 \Longleftrightarrow \epsilon((\pi \otimes \chi_K) \otimes Ad(\tau)) \cdot \chi_K(-1) \cdot \epsilon(\pi) = -1,$$

in which case the Hom space has dimension 1.

Remarks: Consider the case when $\pi = St$ is the Steinberg representation. The representation $\eta^{+-}(\pi)$ is zero and so Thm. 4.4 had better predict that the space $\mathrm{Hom}_{SO(V_K^-)}(\eta^{+-}(\pi), \Sigma_\tau)$ is zero for any τ. Let us check that this is the case. If $\tau \neq \tau \otimes \chi_K$, then one knows that

$$\frac{\chi_K(-1) \cdot \epsilon(St \otimes \chi_K)}{\epsilon(\pi)} = -1 \quad \text{and} \quad \epsilon((St \otimes \chi_K) \otimes \tau \otimes \tau^\vee) = 1.$$

Hence the RHS of the condition on epsilon factors in (ii) is -1, as required. On the other hand, if $\tau = \tau \otimes \chi_K$, then the desired vanishing of the above Hom space is affirmed by (iii).

References

[BFS] S. Bocherer, M. Furusawa and R. Schulze-Pillot, *On the global Gross-Prasad conjecture for Yoshida liftings*, in *Contributions to automorphic forms, geometry, and number theory*, 105–130, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[Co] M. Cognet, *Repr市entation de Weil et changement de base quadratique*, Bull. Soc. Math. France 113 (1985), no. 4, 403–457.

[G] W. T. Gan, *The Saito-Kurokawa space of $PGSp_4$ and its transfer to inner forms*, to appear in *Eisenstein Series and Applications* (proceedings of an AIM workshop), Birkhauser (2007).

[GG] W. T. Gan and N. Gurevich, *Restrictions of Saito-Kurokawa Representations (with an appendix by G. Savin)*, preprint (available at http://www.math.ucsd.edu/ wgan).

[GJR] D. Ginzburg, D. Jiang, S. Rallis, *On the nonvanishing of the central value of the Rankin-Selberg L-functions,* J. Amer. Math. Soc. 17 (2004), no. 3, 679–722.

[GP] B. H. Gross and D. Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}, Canad. J. Math. 44 (1992), no. 5, 974–1002.

[GR] B. H. Gross and M Reeder, *From Laplace to Langlands via representations of orthogonal groups*, Bull. Amer. Math. Soc. (N.S.) 43 (2006), no. 2, 163–205

[HK] M. Harris and S. Kudla, *On a conjecture of Jacquet*, in *Contributions to automorphic forms, geometry, and number theory*, 355–371, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[I] A. Ichino, *Pullbacks of Saito-Kurokawa lifts*, Invent. Math. 162 (2005), no. 3, 551–647.

[II] A. Ichino and T. Ikeda, *On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture*, preprint (preliminary version).

[P1] D. Prasad *Trilinear forms for representations of* GL(2) *and local* ϵ*-factors*, Compositio Math. 75 (1990), no. 1, 1–46.

[P2] D. Prasad *Some applications of seesaw duality to branching laws*, Math. Ann. 304 (1996), no. 1, 1–20.

[PSP] D. Prasad and R. Schulze-Pillot, *Generalized form of a conjecture of Jacquet and a local consequence*, preprint (2006).

[R] B. Roberts, *The non-Archimedean theta correspondence for* GSp(2) *and* GO(4), Trans. Amer. Math. Soc. 351 (1999), no. 2, 781–811.

[Sa] G. Savin , *Restricting small representations of $Sp_4(\mathbb{R})$ to $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$*, appendix to [GG].

[Sch] R. Schimidt, *The Saito-Kurokawa lifting and functoriality*, Amer. J. Math. 127 (2005), no. 1, 209–240.

[W1] J.-L. Waldspurger, *Correspondance de Shimura*, J. Math. Pures et Appl. 59 (1980), 1-133.

[W2] J.-L. Waldspurger, *Correspondance de Shimura et quaternions*, Forum Math. 3 (1991), 219-307.

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