

ON INTEGRAL REPRESENTATIONS OF AUTOMORPHIC L -FUNCTIONS FOR $\mathrm{GSp}(4)$

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ABSTRACT. Among other means, an integral representation gives the meromorphic continuation and the functional equation of an automorphic L -function. In this note, recalling the doubling method as an example, I will try to give some feeling for the techniques. It is often the case that unique models play a role. In the case of $\mathrm{GSp}(4)$, I will review the basic fact about the Whittaker model and the Bessel model.

1. INTRODUCTION

Let F be a number field and \mathbb{A} the ring of adèles of F . Let G be a connected reductive algebraic group over F . Let $L^2_{\mathrm{cusp}}(G)$ denote the space of square integrable functions f on $G(F)\backslash G(\mathbb{A})$ (with fixed central character) such that

$$\int_{U(F)\backslash U(\mathbb{A})} f(ux) du$$

for almost all $x \in G(\mathbb{A})$, where U is the unipotent radical of any proper parabolic subgroup of G over F . Then $G(\mathbb{A})$ acts on $L^2_{\mathrm{cusp}}(G)$ via the right regular representation and $L^2_{\mathrm{cusp}}(G)$ decomposes to a direct sum of irreducible representations of $G(\mathbb{A})$. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, i.e., an irreducible subrepresentation of $L^2_{\mathrm{cusp}}(G)$. Then π is decomposed to a restricted tensor product $\otimes'_v \pi_v$, where π_v is an irreducible admissible representation of $G_v = G(F_v)$ for each place v of F .

For simplicity, we assume that G is split over F . For almost all v , π_v is unramified, i.e., v is non-archimedean and π_v has a fixed vector under the action of $G(\mathfrak{o}_v)$. Let S be a finite set of places of F such that, if $v \notin S$, then π_v is unramified. If $v \notin S$, then we can associate to π_v a semisimple conjugacy class

$$c_v = c_v(\pi_v)$$

in \widehat{G} , where \widehat{G} is the dual group of G over \mathbb{C} . This c_v is called the Satake parameter of π_v .

Let r be a finite dimensional representation of \widehat{G} over \mathbb{C} . If $v \notin S$, then we can define a local factor attached to π_v and r by

$$L_v(s, \pi_v, r) = \det(\mathbf{1} - q_v^{-s} \cdot r(c_v))^{-1}.$$

If v is archimedean, then we can also define a local factor using the Langlands classification. But the definition of $L_v(s, \pi_v, r)$ has not been established in general if v is non-archimedean and $v \in S$. Thus we consider the partial Euler product

$$L^S(s, \pi, r) = \prod_{v \notin S} L_v(s, \pi_v, r)$$

in this note. Langlands proved that this product is absolutely convergent for $\operatorname{Re}(s) \gg 0$.

One of the basic problem in the theory of automorphic forms is to prove the meromorphic continuation and the functional equation of an automorphic L -function. Among other means, there are two important methods: the Langlands-Shahidi method and the Rankin-Selberg method. There is a classification of automorphic L -functions which can be studied using the Langlands-Shahidi method. (The list in [Ki] is quite useful.) The Rankin-Selberg method gives only examples and it becomes harder to find a new integral representation, but there remains a possibility that new ideas come in.

In this note, we give a brief review of the Rankin-Selberg method in the case of $\operatorname{GSp}(4)$, where $\operatorname{GSp}(4)$ is an algebraic group defined by

$$\operatorname{GSp}(4) = \left\{ g \in \operatorname{GL}_4 \mid {}^t g J g = \lambda_g J, \lambda_g \in \operatorname{GL}_1 \right\}, \quad J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix}.$$

Besides $\operatorname{GSp}(4)$ itself, one can use $\operatorname{Sp}(4)$ via the restriction and $\operatorname{SO}(3, 2)$ via the accidental isomorphism to study an automorphic representation of $\operatorname{GSp}(4)$. The dual groups of these groups are given by

$$\widehat{G} = \begin{cases} \operatorname{GSp}(4, \mathbb{C}) & \text{if } G = \operatorname{GSp}(4), \\ \operatorname{SO}(5, \mathbb{C}) & \text{if } G = \operatorname{Sp}(4), \\ \operatorname{Sp}(4, \mathbb{C}) & \text{if } G = \operatorname{SO}(3, 2). \end{cases}$$

Moreover the dual homomorphism of

$$\operatorname{Sp}(4) \rightarrow \operatorname{GSp}(4) \quad (\text{resp. } \operatorname{GSp}(4) \rightarrow \operatorname{PGSp}(4) \simeq \operatorname{SO}(3, 2))$$

is

$$\operatorname{GSp}(4, \mathbb{C}) \rightarrow \operatorname{PGSp}(4, \mathbb{C}) \simeq \operatorname{SO}(5, \mathbb{C}) \quad (\text{resp. } \operatorname{Sp}(4, \mathbb{C}) \rightarrow \operatorname{GSp}(4, \mathbb{C})).$$

We call an automorphic L -function attached to the natural 4-dimensional representation of $\mathrm{GSp}(4, \mathbb{C})$ (resp. the natural 5-dimensional representation of $\mathrm{SO}(5, \mathbb{C})$) the spinor L -function (resp. the standard L -function). Note that one can also use $\mathrm{GL}(4)$ when the functorial lift is available (cf. [AS]), but we do not discuss it in this note.

Remark. For convenience, we describe the isomorphism $\mathrm{PGSp}(4) \simeq \mathrm{SO}(3, 2)$ explicitly. The group $\mathrm{GSp}(4)$ acts on the space of column vectors F^4 on the left. Put

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $\tilde{V} = \wedge^2(F^4)$ is equipped with a non-degenerate symmetric bilinear form (\cdot, \cdot) given by

$$x \wedge y = (x, y) \cdot (e_1 \wedge e_2 \wedge e_3 \wedge e_4)$$

for $x, y \in \tilde{V}$. Put $V = \{x \in \tilde{V} \mid (x, x_0) = 0\}$, where $x_0 = e_1 \wedge e_3 + e_2 \wedge e_4 \in \tilde{V}$. We define a homomorphism $\rho : \mathrm{GSp}(4) \rightarrow \mathrm{SO}(\tilde{V})$ by

$$\rho(g) = \lambda_g^{-1} \cdot \wedge^2(g)$$

for $g \in \mathrm{GSp}(4)$. Since $\rho(g)x_0 = x_0$, this homomorphism induces an exact sequence

$$1 \longrightarrow \mathrm{GL}(1) \xrightarrow{\iota} \mathrm{GSp}(4) \xrightarrow{\rho} \mathrm{SO}(V) \longrightarrow 1.$$

Here $\iota(z) = z \cdot 1_4$ for $z \in \mathrm{GL}(1)$.

2. AN EXAMPLE: THE DOUBLING METHOD

Roughly speaking, the basic identity is of the form

$$(2.1) \quad \int_{H(F) \backslash H(\mathbb{A})} E(h, s) f(h) dh = d^S(s)^{-1} L^S(s + \frac{1}{2}, \pi, r) \prod_{v \in S} Z_v(s)$$

for $\mathrm{Re}(s) \gg 0$. Here $E(h, s)$ is an Eisenstein series, f is a cusp form, $d(s)$ is the normalization factor of $E(h, s)$, and $Z_v(s)$ is a local zeta integral. Since Eisenstein series has the meromorphic continuation and satisfies the functional equation, this identity gives the meromorphic continuation and the functional equation of the automorphic L -function.

The doubling method of Piatetski-Shapiro and Rallis [PSR2] provides the standard L -functions of classical groups. Let $G = \mathrm{Sp}_{2n} \subset \mathrm{GL}_{2n}$, $\tilde{G} = \mathrm{Sp}_{4n} \subset \mathrm{GL}_{4n}$, and $H = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n} \subset \tilde{G}$. Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ and put $\Pi = \pi \otimes \pi^\vee$, where π^\vee is the contragredient representation of π . Then

Π is an irreducible cuspidal automorphic representation of $H(\mathbb{A})$. Let $E(s) \in \text{Ind}_{P(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(|\det|^s)$ be an Eisenstein series on $\tilde{G}(\mathbb{A})$, where P is the Siegel parabolic subgroup of \tilde{G} given by

$$P = \left\{ \begin{pmatrix} a & * \\ 0 & {}_t a^{-1} \end{pmatrix} \in \tilde{G} \mid a \in \text{GL}_{2n} \right\}.$$

It is defined by

$$E(g, s) = \sum_{\gamma \in P \backslash \tilde{G}} \phi(\gamma g, s)$$

for $\text{Re}(s) \gg 0$, where $\phi(g, s)$ is a section of $\text{Ind}_{P(\mathbb{A})}^{\tilde{G}(\mathbb{A})}(|\det|^s)$, and by the meromorphic continuation otherwise. For each $f \in \Pi$, we consider the integral

$$(2.2) \quad \int_{H(F) \backslash H(\mathbb{A})} E(h, s) f(h) dh.$$

First we compute the double coset decomposition

$$\tilde{G} = \coprod_i P \eta_i H$$

and $H_i = H \cap \eta_i^{-1} P \eta_i$ for each i . In this case, except for a unique i , say $i = 0$, H_i contains a unipotent radical of a proper parabolic subgroup of H . Hence we have

$$\begin{aligned} (2.2) &= \int_{H(F) \backslash H(\mathbb{A})} \sum_{\gamma \in P \backslash \tilde{G}} \phi(\gamma h, s) f(h) dh \\ &= \int_{H(F) \backslash H(\mathbb{A})} \sum_i \sum_{\gamma \in H_i \backslash H} \phi(\eta_i \gamma h, s) f(h) dh \\ &= \sum_i \int_{H_i(F) \backslash H(\mathbb{A})} \phi(\eta_i h, s) f(h) dh \\ &= \sum_i \int_{H_i(\mathbb{A}) \backslash H(\mathbb{A})} \int_{H_i(F) \backslash H_i(\mathbb{A})} \phi(\eta_i h h', s) f(h h') dh dh' \end{aligned}$$

for $\text{Re}(s) \gg 0$. Note that the map $\gamma \mapsto \eta_i \gamma$ induces an isomorphism $H_i \backslash H \simeq P \backslash P \eta_i H$. Since f is cuspidal, we obtain

$$\begin{aligned} (2.2) &= \int_{H_0(\mathbb{A}) \backslash H(\mathbb{A})} \int_{H_0(F) \backslash H_0(\mathbb{A})} \phi(\eta_0 h h', s) f(h h') dh dh' \\ &= \int_{H_0(\mathbb{A}) \backslash H(\mathbb{A})} \phi(\eta_0 h', s) \int_{H_0(F) \backslash H_0(\mathbb{A})} (\Pi(h') f)(h) dh dh'. \end{aligned}$$

Here we have used the fact that $\eta_0 h \eta_0^{-1} \in P(\mathbb{A})$ and

$$\phi(\eta_0 h h', s) = \phi(\eta_0 h', s)$$

for $h \in H_0(\mathbb{A})$. We remark that

$$H_0 = \{(h, h) \mid h \in \mathrm{Sp}_{2n}\} \subset H = \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n}$$

if η_0 is suitably chosen.

Next we consider the functional l on Π given by the integral

$$l(f) = \int_{H_0(F) \backslash H_0(\mathbb{A})} f(h) dh$$

for $f \in \Pi$. It is obvious that l is an element in

$$\mathrm{Hom}_{H_0(\mathbb{A})}(\Pi, \mathbb{C}).$$

Moreover it is well-known that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{H_{0,v}}(\Pi_v, \mathbb{C}) = 1$$

for all places v of F . In fact, since $\Pi_v = \pi_v \otimes \pi_v^{\vee}$, this follows from the uniqueness of invariant pairings between π_v and π_v^{\vee} . Hence there exist elements l_v in $\mathrm{Hom}_{H_{0,v}}(\Pi_v, \mathbb{C})$ such that

$$l(f) = \prod_v l_v(f_v)$$

for $f = \otimes'_v f_v \in \Pi = \otimes'_v \Pi_v$. We now return to the computation of (2.2). We may assume that f and ϕ are decomposable. Then we have

$$\int_{H_0(F) \backslash H_0(\mathbb{A})} (\Pi(h')f)(h) dh = l(\Pi(h')f) = \prod_v l_v(\Pi_v(h'_v)f_v)$$

for $h' = (h'_v) \in H(\mathbb{A})$. We also have

$$\phi(g, s) = \prod_v \phi_v(g_v, s)$$

for $g = (g_v) \in \tilde{G}(\mathbb{A})$. Thus we obtain

$$(2.2) = \prod_v \int_{H_{0,v} \backslash H_v} \phi_v(\eta_0 h'_v, s) l_v(\Pi_v(h'_v)f_v) dh'_v.$$

It remains to study the local integral

$$Z_v(s) = \int_{H_{0,v} \backslash H_v} \phi_v(\eta_0 h'_v, s) l_v(\Pi_v(h'_v)f_v) dh'_v.$$

A crucial step is to compute $Z_v(s)$ when everything is unramified. In this case, the answer is

$$Z_v(s) = d_v(s)^{-1} L_v(s + \frac{1}{2}, \pi_v, \mathrm{st}),$$

where

$$d_v(s) = \zeta_v(s + n + \frac{1}{2}) \prod_{i=1}^n \zeta_v(2s + 2n + 1 - 2i)$$

and st is the standard representation of $\text{SO}(2n + 1, \mathbb{C})$ on \mathbb{C}^{2n+1} . Thus we obtain the identity (2.1) for $r = \text{st}$.

Finally, we study the local zeta integrals $Z_v(s)$ in general (cf. [KR1], [KR2], [HKS], [LR]). We write $E(s) = E(s, \phi)$ and $Z_v(s) = Z_v(s, \phi_v)$ to emphasize the dependence on ϕ and ϕ_v , respectively. Recall that $Z_v(s)$ is defined by an integral which is convergent only for $\text{Re}(s) \gg 0$. To study the analytic properties of $L^S(s, \pi, \text{st})$, one needs to prove the meromorphic continuation of $Z_v(s)$ and the non-vanishing of $Z_v(s)$. For each place $v \in S$, a candidate $L_v^{\text{PSR}}(s, \pi_v, \text{st})$ of the local L -factor is given by the greatest common denominator of $Z_v(s - \frac{1}{2}, \phi_v)$ when ϕ_v runs over good sections (at least if v is non-archimedean). To prove the functional equation of $L^S(s, \pi, \text{st})$, we first prove the local functional equation

$$Z_v(-s, M_v(s)\phi_v) = \Gamma_v(s) \cdot Z_v(s, \phi_v),$$

where $M_v(s) : \text{Ind}_{P_v}^{\tilde{G}_v}(|\det|^s) \rightarrow \text{Ind}_{P_v}^{\tilde{G}_v}(|\det|^{-s})$ is the intertwining operator and $\Gamma_v(s)$ is a meromorphic function. Let $\gamma_v^{\text{PSR}}(s, \pi_v, \text{st}, \psi_v)$ be a certain modification of $\Gamma_v(s - \frac{1}{2})$, which is expected to coincide with the local γ -factor

$$\gamma_v(s, \pi_v, \text{st}, \psi_v) = \varepsilon_v(s, \pi_v, \text{st}, \psi_v) \cdot \frac{L_v(1 - s, \pi_v^\vee, \text{st})}{L_v(s, \pi_v, \text{st})}.$$

Indeed, $\gamma_v^{\text{PSR}}(s, \pi_v, \text{st}, \psi_v)$ satisfies the Ten Commandments of γ -factors. Then the functional equation

$$L^S(s, \pi, \text{st}) = \prod_{v \in S} \gamma_v^{\text{PSR}}(s, \pi_v, \text{st}, \psi_v) \cdot L^S(1 - s, \pi^\vee, \text{st})$$

follows from the basic identity (2.1) and the global functional equation

$$E(-s, M(s)\phi) = E(s, \phi)$$

of the Eisenstein series. See [LR] for more details.

3. WHITTAKER MODELS

In this section, we introduce a certain class of unique models. This model can be defined when G is quasi-split over F and called the Whittaker model.

Let us begin with the case $G = \text{GL}_2$. Let $\pi = \otimes'_v \pi_v$ be an irreducible cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$. We fix a non-trivial

additive character ψ of \mathbb{A}/F . For each $f \in \pi$, define a function W_f on $G(\mathbb{A})$ by

$$W_f(g) = \int_{U(F)\backslash U(\mathbb{A})} f(ug)\chi(u)^{-1} du$$

for $g \in G(\mathbb{A})$. Here

$$U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in G \right\}$$

and

$$\chi \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = \psi(x)$$

for $x \in \mathbb{A}$. Note that χ is a character of $U(F)\backslash U(\mathbb{A})$. It is well-known that W_f is not identically zero if $f \neq 0$. This function W_f is called a Whittaker function and is a kind of the Fourier coefficients of classical modular forms. The map $f \mapsto W_f$ belongs to the space

$$\text{Hom}_{G(\mathbb{A})}(\pi, \text{Ind}_{U(\mathbb{A})}^{G(\mathbb{A})}(\chi)),$$

where $\text{Ind}_{U(\mathbb{A})}^{G(\mathbb{A})}(\chi)$ is the space of smooth functions W on $G(\mathbb{A})$ such that

$$W(ug) = \chi(u)W(g)$$

for $u \in U(\mathbb{A})$ and $g \in G(\mathbb{A})$, and $G(\mathbb{A})$ acts on $\text{Ind}_{U(\mathbb{A})}^{G(\mathbb{A})}(\chi)$ via the right regular representation. Moreover the local component of this space of homomorphisms satisfies a uniqueness, i.e.,

$$\dim_{\mathbb{C}} \text{Hom}_{G_v}(\pi_v, \text{Ind}_{U_v}^{G_v}(\chi_v)) = 1.$$

To be precise, if v is archimedean, then one has to impose a moderate growth condition on $\text{Ind}_{U_v}^{G_v}(\chi_v)$. Hence there exist elements W_{f_v} in $\text{Ind}_{U_v}^{G_v}(\chi_v)$ such that

$$W_f(g) = \prod_v W_{f_v}(g_v)$$

for $f = \otimes'_v f_v \in \pi = \otimes'_v \pi_v$ and $g = (g_v) \in G(\mathbb{A})$. This decomposability plays a role as in §2.

For general G , we take U to be a unipotent radical of a Borel subgroup of G over F and χ a non-degenerate character of $U(F)\backslash U(\mathbb{A})$. Here a character χ of $U(\mathbb{A})$ is called non-degenerate if it is non-trivial on the one-parameter subgroup attached to any simple root of U . In the case $G = \text{GSp}(4)$, we can take

$$U = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \in G \right\}$$

and

$$\chi \left(\begin{pmatrix} 1 & x & * & * \\ 0 & 1 & * & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \right) = \psi(x + y),$$

where ψ is a fixed non-trivial additive character of \mathbb{A}/F . Note that the coordinate x (resp. y) corresponds to the simple root $e_1 - e_2$ (resp. $2e_2$). Let $\pi = \otimes'_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$. For each $f \in \pi$, define a function W_f on $G(\mathbb{A})$ by

$$W_f(g) = \int_{U(F) \backslash U(\mathbb{A})} f(ug) \chi(u)^{-1} du$$

for $g \in G(\mathbb{A})$. In this case, W_f may be identically zero for all $f \in \pi$. We call π globally generic if there exists $f \in \pi$ such that W_f is not identically zero. Shalika [Sh] (see also [Wa] in the archimedean case) proved that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{U_v}^{G_v}(\chi_v)) \leq 1.$$

If the equality holds above, then π_v is said to be generic. Hence if π is globally generic, then π_v is generic for all places v of F and there exist elements W_{f_v} in $\mathrm{Ind}_{U_v}^{G_v}(\chi_v)$ such that

$$W_f(g) = \prod_v W_{f_v}(g_v)$$

for $f = \otimes'_v f_v \in \pi = \otimes'_v \pi_v$ and $g = (g_v) \in G(\mathbb{A})$.

As an example of integral representations which use the Whittaker model, we recall Novodvorsky's integral [No] of the spinor L -functions $L(s, \pi)$ for GSp_4 . For $f = \otimes'_v f_v \in \pi$, the basic identity is of the form

$$\int_{F^\times \backslash \mathbb{A}^\times} \int_{(F \backslash \mathbb{A})^3} f(u(x)t(y)) \psi(x_1)^{-1} |y|^{s-1/2} dx d^\times y = L^S(s, \pi) \prod_{v \in S} Z_v(s)$$

for $\mathrm{Re}(s) \gg 0$, where

$$u(x) = \begin{pmatrix} 1 & x_1 & x_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & x_3 & -x_1 & 1 \end{pmatrix}, \quad t(y) = \begin{pmatrix} y & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$Z_v(s) = \int_{F_v^\times} \int_{F_v} W_{f_v} \left(\begin{pmatrix} y_v & 0 & 0 & 0 \\ 0 & y_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & x_v & 0 & 1 \end{pmatrix} \right) |y_v|^{s-3/2} dx_v d^\times y_v.$$

4. BESSEL MODELS

In this section, we introduce a different class of unique model. This model can be defined for special orthogonal groups or unitary groups. Note that $\mathrm{PGSp}(4) \simeq \mathrm{SO}(3, 2)$.

Let $G = \mathrm{GSp}(4)$ and U the unipotent radical of the Siegel parabolic subgroup of G given by

$$U = \left\{ \begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix} \mid b \in M_2, {}^t b = b \right\}.$$

We fix $\beta \in \mathrm{GL}_2$ such that ${}^t \beta = \beta$ and define a character ψ_β on $U(F) \backslash U(\mathbb{A})$ by

$$\psi_\beta \left(\begin{pmatrix} 1_2 & b \\ 0 & 1_2 \end{pmatrix} \right) = \psi(\mathrm{tr}(\beta b)),$$

where ψ is a fixed non-trivial additive character of \mathbb{A}/F . But $\mathrm{Ind}_{U(\mathbb{A})}^{G(\mathbb{A})}(\psi_\beta)$ is too big to ensure the uniqueness.

Let T be the identity component of the similitude orthogonal group of β given by

$$T = \{h \in \mathrm{GL}_2 \mid {}^t h \beta h = \det(h) \beta\}.$$

For simplicity, we assume that T is not split over F . We identify T with a subgroup of G by the embedding

$$h \longmapsto \begin{pmatrix} h & 0 \\ 0 & \det(h) {}^t h^{-1} \end{pmatrix}.$$

Then T acts on U via the adjoint action and we have

$$\psi_\beta(t^{-1} u t) = \psi_\beta(u)$$

for $t \in T(\mathbb{A})$ and $u \in U(\mathbb{A})$. More explicitly, write

$$\beta = \begin{pmatrix} \beta_1 & \beta_2/2 \\ \beta_2/2 & \beta_3 \end{pmatrix}$$

and put $\Delta = -4 \det(\beta) = \beta_2^2 - 4\beta_1\beta_3 \in F^\times$. Then our assumption on T becomes $\Delta \notin F^{\times, 2}$. Let $E = F(\delta)$ be the quadratic extension of F with $\delta^2 = \Delta$ and put

$$A_\delta = \begin{pmatrix} \beta_2 & 2\beta_3 \\ -2\beta_1 & -\beta_2 \end{pmatrix}.$$

Then an F -algebra homomorphism

$$x \cdot 1_2 + y \cdot A_\delta \longmapsto x + y\delta$$

induces an isomorphism $T = F(A_\delta)^\times \simeq E^\times$.

Instead of U , we consider the subgroup R of G given by

$$R = TU.$$

Let ν be a character of $\mathbb{A}_E^\times/E^\times$ and regard it as a character of $T(\mathbb{A})$. Put

$$\chi(tu) = \nu(t)\psi_\beta(u)$$

for $t \in T(\mathbb{A})$ and $u \in U(\mathbb{A})$. Then χ is a character of $R(F)\backslash R(\mathbb{A})$. For each $f \in \pi$, define a function B_f on $G(\mathbb{A})$ by

$$B_f(g) = \int_{\mathbb{A}^\times R(F)\backslash R(\mathbb{A})} f(rg)\chi(r)^{-1} dr$$

for $g \in G(\mathbb{A})$. Here we add the assumption that $\nu|_{\mathbb{A}^\times} = \omega_\pi$, where ω_π is the central character of π , so that this integral is well-defined. The map $f \mapsto B_f$ belongs to the space

$$\mathrm{Hom}_{G(\mathbb{A})}(\pi, \mathrm{Ind}_{R(\mathbb{A})}^{G(\mathbb{A})}(\chi)).$$

It is known that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G_v}(\pi_v, \mathrm{Ind}_{R_v}^{G_v}(\chi_v)) \leq 1,$$

but I could not find a reference for the proof. Admitting this uniqueness, there exist elements B_{f_v} in $\mathrm{Ind}_{R_v}^{G_v}(\chi_v)$ such that

$$B_f(g) = \prod_v B_{f_v}(g_v)$$

for $f = \otimes'_v f_v \in \pi = \otimes'_v \pi_v$ and $g = (g_v) \in G(\mathbb{A})$.

As an example of integral representations which use the Bessel model, we recall Andrianov's integral [PS] of twists $L(s, \pi, \mu)$ of the spinor L -functions for GSp_4 , where μ is a character of $\mathbb{A}^\times/F^\times$. Let E be a quadratic extension of F , σ the non-trivial automorphism of E/F , and ν a character of $\mathbb{A}_E^\times/E^\times$. Put

$$H = \{h \in \mathrm{GL}_2(E) \mid \det(h) \in F^\times\}.$$

Let $E(s) \in \mathrm{Ind}_{B(\mathbb{A})}^{H(\mathbb{A})}(\chi\delta_B^{s/2})$ be an Eisenstein series on $H(\mathbb{A})$, where B is the standard Borel subgroup of H , χ is a character of $B(\mathbb{A})$ defined by

$$\chi\left(\begin{pmatrix} xa^\sigma & * \\ 0 & a \end{pmatrix}\right) = \mu(x)\nu(a)^{-1}$$

for $x \in \mathbb{A}^\times$ and $a \in \mathbb{A}_E^\times$, and δ_B is the modulus character of $B(\mathbb{A})$. For $f = \otimes'_v f_v \in \pi$ and a decomposable section ϕ of $\mathrm{Ind}_{B(\mathbb{A})}^{H(\mathbb{A})}(\chi\delta_B^{s/2})$, the basic identity is of the form

$$\int_{\mathbb{A}^\times H(F)\backslash H(\mathbb{A})} E(h, s)f(h) dh = d^S(s)^{-1}L^S(s + \frac{1}{2}, \pi, \mu) \prod_{v \in S} Z_v(s)$$

for $\operatorname{Re}(s) \gg 0$, where $d(s) = L(s + 1, \mu \cdot \nu|_{\mathbb{A}^\times})$ and

$$Z_v(s) = \int_{B'(F_v) \backslash H(F_v)} \phi_v(h_v, s) B_{f_v}(h_v) dh_v$$

with

$$B' = \left\{ \begin{pmatrix} a^\sigma & * \\ 0 & a \end{pmatrix} \in H \mid a \in E^\times \right\}.$$

REFERENCES

For the generality of automorphic L -functions, Borel's article [Bo] is recommended. Bump's article [Bu2] gives a nice survey of the Rankin-Selberg method.

Doubling method. The basic reference is [PSR2]. For the classical setting, see also [Ga], [Bö1], [Bö2]. In [KR2], Kudla and Rallis studied the poles of the standard L -functions. In the following articles, the local zeta integrals and the local factors have been studied.

- [KR1] analysis of local zeta integrals
- [PSR1] existence of local γ -factors
- [LR] basic properties of local γ -factors
- [RS] stability of local γ -factors

Whittaker model. In [Sh], Shalika proved the uniqueness of the Whittaker model in general. See also [Wa] in the archimedean case. Explicit formulas for the Whittaker functions have been established in the following cases.

- [CS] unramified representations (the Casselman-Shalika formula)
- [Li] some Iwahori-fixed vectors in unramified principal series
- [Re] Iwahori-fixed vectors in unramified principal series
- [Od] large discrete series of $\operatorname{Sp}_4(\mathbb{R})$
- [MO] P_J -principal series of $\operatorname{Sp}_4(\mathbb{R})$
- [Mo3] P_J -principal series of $\operatorname{Sp}_4(\mathbb{R})$ (different expression from [MO])
- [Ni2] class-one representations of $\operatorname{Sp}_4(\mathbb{R})$
- [Pr] class-one representations of $\operatorname{Sp}_4(\mathbb{C})$
- [Is2] principal series of $\operatorname{Sp}_4(\mathbb{R})$

The basic reference for Novodvorsky's integral of the spinor L -functions for GSp_4 is [No]. Bump's article [Bu1] is also useful. The local zeta integrals have been studied in the following cases.

- [TB] over p -adic fields
- [Mo4] large discrete series and P_J -principal series of $\operatorname{GSp}_4(\mathbb{R})$

The following is a list of papers on integral representations which use the Whittaker model.

- [No] Novodvorsky's integral of L -functions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$
- [So1] local L and γ -factors for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ over p -adic fields
- [Mo3] some local zeta integrals for $\mathrm{GSp}_4(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{R})$
- [Gi2] adjoint L -functions for Sp_4 (degree 10)
- [Ji] L -functions for $\mathrm{GSp}_4 \times \mathrm{GSp}_4$ (degree 16)
- [GRS] L -functions for $\mathrm{Sp}_{2n} \times \mathrm{GL}_k$ (Shimura integral)
- [BFG] spinor L -functions and standard L -functions (two variables)

If the central character is trivial, we can use integral representations of L -functions for SO_5 .

- [GPS] L -functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_k$ ($k = n, n + 1$)
- [Gi1] L -functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_k$ ($k \leq n$)
- [So2] L -functions for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_k$ ($k > n$)
- [So2] local γ -factors for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_k$ over p -adic fields
- [So3] local γ -factors for $\mathrm{SO}_{2n+1} \times \mathrm{GL}_k$ over \mathbb{R} and \mathbb{C}

Bessel model. The uniqueness of the Bessel model was proved in [NPS], [Ro] in some cases, but I could not find a reference for the proof in general. Explicit formulas for the Bessel functions have been established in the following cases.

- [Su] unramified representations of Sp_4
- [BFF] unramified representations (the Casselman-Shalika like formula)
- [Ni1] class-one representations of $\mathrm{Sp}_4(\mathbb{R})$
- [Mi1] large discrete series and P_J -principal series of $\mathrm{Sp}_4(\mathbb{R})$
- [Mi2] some derived functor modules of $\mathrm{Sp}_4(\mathbb{R})$
- [Is1] principal series of $\mathrm{Sp}_4(\mathbb{R})$

Besides the original article [An], the basic reference for Andrianov's integral of the spinor L -functions for GSp_4 is [PS]. See also [Su] for the L -functions of degree 4 for inner forms of GSp_4 . The local zeta integrals have been studied in the following cases.

- [PSS] over p -adic fields
- [Ho] class-one representation of $\mathrm{Sp}_4(\mathbb{R})$
- [Mi1] large discrete series and P_J -principal series of $\mathrm{Sp}_4(\mathbb{R})$

In [Fu], using the Bessel model, Furusawa constructed an integral representation of L -functions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

Others. In [GPSR], Ginzburg, Piatetski-Shapiro, and Rallis found an integral representation of the L -function for $\mathrm{SO}_n \times \mathrm{GL}_k$ in general. They used the so-called "Whittaker-Shintani model" (or "generalized Bessel model" or "generalized Gelfand-Graev model"). Note that, in [MS2], Murase and Sugano also constructed an integral representation of the standard L -function for some special orthogonal groups using the

Shintani model. Explicit formulas for the Whittaker-Shintani functions have been established in the following cases.

- [KMS] unramified representations
- [Mo1] some representations of $(\mathrm{Sp}_4(\mathbb{R}), \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}))$
- [Mo2] some representations of $(\mathrm{Sp}_4(\mathbb{R}), \mathrm{SL}_2(\mathbb{C}))$

In [MS1], using the Fourier-Jacobi model, Murase and Sugano constructed an integral representation of the standard L -functions for Sp_{2n} . The uniqueness of the Fourier-Jacobi model for representations of Sp_4 and GSp_4 was proved in [BR] and [AP], respectively. In the classical setting, Kohnen and Skoruppa [KS] considered certain Dirichlet series associated to the Fourier-Jacobi coefficients which give the spinor L -functions for GSp_4 . Also, in [He] and [BH], Heim, and Böcherer and Heim found an integral representation of L -functions for $\mathrm{GSp}_4 \times \mathrm{GL}_2$ in which the Fourier-Jacobi coefficients play a role. Explicit formulas for the Fourier-Jacobi type spherical functions have been established in the following cases.

- [Mu] unramified representations of Sp_4
- [Hi1] discrete series of $\mathrm{Sp}_4(\mathbb{R})$
- [Hi2] P_J -principal series of $\mathrm{Sp}_4(\mathbb{R})$
- [Hi3] principal series of $\mathrm{Sp}_4(\mathbb{R})$

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