ARCHIMEDEAN ZETA INTEGRALS ON $GSp(4)$

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Introduction

When we study automorphic L-functions via integral representations, we face to the difficulty of treatment of the local zeta integrals at bad places. In this note we report recent progress on the explicit computation of the local zeta integrals at real places for generic cusp forms on $GSp(4)$.

Let $\Pi = \otimes_v' \Pi_v$ be an automorphic cuspidal representation of $GSp(4, \mathbf{A})$. Here $\mathbf{A}$ is the adele of $\mathbf{Q}$. We assume Π is *generic*, that is, Π has a global Whittaker model. Then according to the result of Kostant [Ko] and Vogan [V], the representation Π_∞ of $G = GSp(4, \mathbf{R})$ is equivalent to one of the following ([Mo-3]):

(1) (limit of) large discrete series,
(2) generalized principal series induced from Jacobi parabolic subgroup,
(3) generalized principal series induced from Siegel parabolic subgroup,
(4) principal series induced from minimal parabolic subgroup.

Novodvorsky [N] considered integral representations of spinor L-functions $L_1(s, \Pi)$ on $GSp(4)$ which works for generic cusp forms. Moriyama [Mo-2] (the cases (1) and (2)) and Moriyama and the author [Is-Mo] (the case of (4)) computed the archimedean part of Novodvorsky's zeta integrals by using explicit formulas of Whittaker functions on $GSp(4, \mathbf{R})$, which have been developed by Oda, Miyazaki, Niwa, Hirano, Moriyama, Ishii, etc (see Ichino's references [Ic]). At the present explicit formulas of Whittaker functions for the case (3) have not yet known.

On the other hand, as for the standard L-function $L_2(s, \Pi)$, some integral representations are known for generic cusp forms ([B-F-G], [G-R-S]). We give examples of the computation in case of (2) and (4).

1. Representations of $GSp(4, \mathbf{R})$ and local L and ε factors

We list the local L and ε factors determined by the Langlands parameter of Π_∞. We use the subscript $_1$ (resp. $_2$) for the spinor (resp. standard) L-functions. For the case of the discrete series, see [Mo-3] for the precise. Note that there is a difference in the notation between [Mo-3] and ours. We denote by $\lambda = (\lambda_1, \lambda_2)$ the Blattner parameter of the discrete series π_λ.

(1) (limit of) large discrete series $\pi_\lambda[\omega]$ $(1 - \lambda_1 \le \lambda_2 \le 0)$.

- $\begin{cases} L_1(s, \Pi_\infty) = \Gamma_{\mathbf{C}}(s + \frac{\omega+\lambda_1-\lambda_2-1}{2})\Gamma_{\mathbf{C}}(s + \frac{\omega+\lambda_1+\lambda_2-1}{2}); \\ \varepsilon_1(s, \Pi_\infty, \psi_\infty), = (-1)^{\lambda_1} \end{cases}$
- $\begin{cases} L_2(s, \Pi_\infty) = \Gamma_{\mathbf{R}}(s+1)\Gamma_{\mathbf{C}}(s + \lambda_1 - 1)\Gamma_{\mathbf{C}}(s - \lambda_2); \\ \varepsilon_2(s, \Pi_\infty, \psi_\infty) = (\sqrt{-1})^{2\lambda_1-2\lambda_2+1}. \end{cases}$

(2) P_J-principal series $I(P_J;\sigma_{k,\pm},\nu)$

The data for P_J-principal series is as follows. Let $P_J = M_J A_J N_J$ be the Langlands decomposition of Jacobi parabolic subgroup of $G = GSp(4,\mathbf{R})$, where

$$M_J = \{ \left(\begin{array}{cc|cc} \varepsilon_0 & & & \\ & \varepsilon_0 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \cdot \left(\begin{array}{cc|cc} \varepsilon_1 & & & \\ & a & & b \\ \hline & & \varepsilon_1 & \\ & c & & d \end{array}\right) \mid \varepsilon_0, \varepsilon_1 \in \{\pm 1\}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{R})\},$$

$$A_J = \{a = a_0 \operatorname{diag}(a_1, 1, a_1^{-1}, 1) \mid a_i > 0 \ (i = 0,1)\},$$

$$N_J = \{n(n_0, n_1, n_2, 0) \in G\},$$

with

$$n(n_0,n_1,n_2,n_3) := \left(\begin{array}{cc|cc} 1 & & n_1 & n_2 \\ & 1 & n_2 & n_3 \\ \hline & & 1 & \\ & & & 1 \end{array}\right) \cdot \left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array}\right).$$

Define a representation $\sigma_{k,\pm}$ of Levi part M_J such that $\sigma_{k,\pm}|_{SL(2,\mathbf{R})} = D_k \oplus D_{-k}$ and $\sigma_{k,\pm}(\operatorname{diag}(-1,1,-1,1)) = \pm 1$, with D_k the discrete series representation of $SL(2,\mathbf{R})$ with Blattner parameter k. For $\nu = (\nu_1, \omega) \in \operatorname{Lie}(A_J)^*_{\mathbf{C}} \cong \mathbf{C}^2$, let $\exp(\nu)$ be the quasi character of A_J defined by $\exp(\nu)(a_0 \operatorname{diag}(a_1, 1, a_1^{-1}, 1)) = a_0^{\omega} a_1^{\nu_1}$.

The P_J-principal series $I(P_J;\sigma_{k,\pm},\nu)$ is the induced representation $\operatorname{Ind}_{P_J}^{G}(\sigma_{k,\pm} \otimes \exp(\nu + \rho_J) \otimes 1_{N_J})$. The local L and ε factors are

- $\begin{cases} L_1(s,\Pi_\infty) = \Gamma_{\mathbf{C}}(s + \frac{\omega+\nu_1+k-1}{2})\Gamma_{\mathbf{C}}(s + \frac{\omega-\nu_1+k-1}{2}); \\ \varepsilon_1(s,\Pi_\infty,\psi_\infty) = (-1)^k, \end{cases}$
- $\begin{cases} L_2(s,\Pi_\infty) = \Gamma_{\mathbf{R}}(s+1)\Gamma_{\mathbf{R}}(s+\nu_1)\Gamma_{\mathbf{R}}(s-\nu_1)\Gamma_{\mathbf{C}}(s+k-1); \\ \varepsilon_2(s,\Pi_\infty,\psi_\infty) = (-1)^k. \end{cases}$

(3) principal series $I(P_0;\sigma,\nu)$.

Let $P_0 = MAN$ be a Langlands decomposition of minimal parabolic subgroup P_0 of G, where

$$M = \langle \gamma_0 := \operatorname{diag}(-1,-1,1,1),\ \gamma_1 := \operatorname{diag}(-1,1,-1,1),\ \gamma_2 := \operatorname{diag}(1,-1,1,-1)\rangle,$$

$$A = \{a = \operatorname{diag}(a_0 a_1, a_0 a_2, a_1^{-1}, a_2^{-1}) \mid a_i > 0 \ (i = 0,1,2)\},$$

$$N = \{n(n_0,n_1,n_2,n_3) \in G\}.$$

For a character $\sigma \in \widehat{M}$, define $\delta_I \in \{0,1\}$ $(I \subset \{0,1,2\})$ by $\sigma(\prod_{i\in I}\gamma_i) = (-1)^{\delta_I}$. For $\nu = (\nu_0,\nu_1,\nu_2) \in \mathbf{C}^3$, define a quasi character $\exp(\nu)$ of A by $\exp(\nu)(a(a_0,a_1,a_2)) = a_0^{\nu_0} a_1^{\nu_1} a_2^{\nu_2}$ and set $\omega = 2\nu_0 - \nu_1 - \nu_2$. We call the induced representation $I(P_0;\sigma,\nu) = \operatorname{Ind}_{P_0}^{G}(\sigma \otimes \exp(\nu+\rho) \otimes 1_N)$ the principal series representation. Here $\rho = (3/2, 2, 1)$.

- $\begin{cases} L_1(s,\Pi_\infty) = \Gamma_{\mathbf{R}}(s + \frac{\omega+\nu_1+\nu_2}{2} + \delta_{\{0\}})\Gamma_{\mathbf{R}}(s + \frac{\omega-\nu_1+\nu_2}{2} + \delta_{\{0,1\}}) \\ \qquad\qquad\qquad \times \Gamma_{\mathbf{R}}(s + \frac{\omega+\nu_1-\nu_2}{2} + \delta_{\{0,2\}})\Gamma_{\mathbf{R}}(s + \frac{\omega-\nu_1-\nu_2}{2} + \delta_{\{0,1,2\}}); \\ \varepsilon_1(s,\Pi_\infty,\psi_\infty) = (\sqrt{-1})^{\delta_{\{0\}}+\delta_{\{0,1\}}+\delta_{\{0,2\}}+\delta_{\{0,1,2\}}}, \end{cases}$

- $$\begin{cases} L_2(s,\Pi_\infty) = \Gamma_{\mathbf{R}}(s+1)\Gamma_{\mathbf{R}}(s+\nu_1+\delta_{\{1\}})\Gamma_{\mathbf{R}}(s-\nu_1+\delta_{\{1\}}) \\ \qquad\qquad\qquad \times\Gamma_{\mathbf{R}}(s+\nu_2+\delta_{\{2\}})\Gamma_{\mathbf{R}}(s-\nu_2+\delta_{\{2\}}); \\ \varepsilon_2(s,\Pi_\infty,\psi_\infty) = (-1)^{\delta_{\{1\}}+\delta_{\{2\}}}. \end{cases}$$

2. EXPLICIT FORMULAS OF WHITTAKER FUNCTIONS ON $Sp(4,\mathbf{R})$

For an irreducible admissible representation π of $G_0 := Sp(4,\mathbf{R})$, and a non-degenerate unitary character η of the maximal unipotent subgroup N of G_0 consider the intertwining space

$$\mathrm{Wh}(\pi) = \mathrm{Hom}_{(\mathfrak{g}_0,K_0)}(\pi, C^\infty_\eta(N\backslash G_0)).$$

Here $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$, $K_0 = K\cap G_0$ and

$$C^\infty_\eta(N\backslash G_0) = \{f\in C^\infty(G_0,\mathbf{C}) \mid f(ng)=\eta(n)f(g),\ \ \forall(n,g)\in N\times G_0\}.$$

For a nonzero intertwiner $\Phi\in\mathrm{Wh}(\pi)$ and a vector $v\in\pi$, we call $W_v=\Phi(v)$ the Whittaker function corresponding to v. To find explicit formula of W_v, we derive certain system of partial differential equations for W_v from a characterization of π. The system becomes a holonomic system of rank 4 or 8 in our situation, and among the solutions of the system we need a Whittaker function which is of moderate growth. Some kinds of integral representation of moderate growth Whittaker functions have been obtained, especially Mellin-Barnes type integral representations are useful for the computation of zeta integrals.

Because of the Iwasawa decomposition $G_0 = NA_0K_0$, W_v can be determined by its restriction $W_v|_{A_0}$. Here $A_0 = A\cap G_0$. Set $a = \mathrm{diag}(a_1,a_2,a_1^{-1},a_2^{-1})\in A_0$ and we give Mellin-Barnes integral representations of $W_v(a)$.

The maximal compact subgroup $K_0 = Sp(4,\mathbf{R})\cap O(4)$ of $Sp(4,\mathbf{R})$ is isomorphic to the unitary group $U(2)$ of degree two:

$$U(2)\ni A+\sqrt{-1}B \mapsto \begin{pmatrix} A & B \\ -B & A\end{pmatrix}\in K_0,\quad (A,B\in M(2,\mathbf{R})).$$

Then

$$\widehat{K_0} = \{\tau_{(\lambda_1,\lambda_2)} := \mathrm{sym}^{\lambda_1-\lambda_2}\otimes\det{}^{\lambda_2} \mid \lambda_1,\lambda_2\in\mathbf{Z},\ \lambda_1\geq\lambda_2\}.$$

We take the standard basis $\{v_k \mid 0\leq k\leq d=\lambda_1-\lambda_2\}$ of $\tau_{(\lambda_1,\lambda_2)}$ (see [O] etc. for the action of K_0 on the standard basis).

We give examples of Mellin-Barnes integral representations for Whittaker functions W_v.

(1) (limit of) large discrete series $\pi_{(-\lambda_2,-\lambda_1)}$ $(1-\lambda_1\leq\lambda_2\leq 0)$ ([O], [Mo-1], [Mo-2]).

For $v_k\in\tau_{(-\lambda_2,-\lambda_1)}$ = minimal K_0-type of $\pi_{(-\lambda_2,-\lambda_1)}$ $(0\leq k\leq\lambda_1-\lambda_2)$ we have

$$W_{v_k}(a) = (2\pi i)^{-k}\binom{d}{k}a_1^{\lambda_1-k+1}a_2^{\lambda_2+k}\exp(-2\pi a_2^2)\int_{-i\infty}^{i\infty}\frac{ds_1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{ds_2}{2\pi i}\Big(\pi\frac{a_1}{a_2}\Big)^{-2s_1}(4\pi a_2^2)^{-s_2}$$
$$\cdot(2s_1)_k\,\Gamma(s_1)\Gamma(s_1-s_2)\Gamma\Big(s_2+\frac{1}{2}\Big)\Gamma\Big(s_2-\lambda_2+\frac{1}{2}\Big).$$

Here $(a)_n = \Gamma(a+n)/\Gamma(a)$.

(2) $I(P_J;\sigma_{k,+},\nu)$ ([M-O2], [Mo-2]).

For the 'corner' vector $v_0 \in \tau_{(k,k)}$ we have

$$W_{v_0}(a) = a_1^{k+1} a_2^k \exp(-2\pi a_2^2) \int_{-i\infty}^{i\infty} \frac{ds_1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{ds_2}{2\pi i} \Big(\pi \frac{a_1}{a_2}\Big)^{-2s_1} (4\pi a_2^2)^{-s_2}$$
$$\cdot \Gamma(s_1)\Gamma(s_1 - s_2)\Gamma\Big(s_2 + \frac{-k+\nu_1+1}{2}\Big)\Gamma\Big(s_2 + \frac{-k-\nu_1+1}{2}\Big).$$

(3) class one principal series $I(P_0; \mathrm{triv}, \nu)$ ([M-O1],[Ni],[Is],[Is-Mo]).

For the spherical vector $v_0 \in \tau_{(0,0)}$ we have

$$W_{v_0}(a) = a_1^2 a_2 \int_{-i\infty}^{i\infty} \frac{ds_1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{ds_1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dt_1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{dt_2}{2\pi i} \Big(\pi \frac{a_1}{a_2}\Big)^{-2s_1} (\pi a_2^2)^{-s_2}$$
$$\cdot \Gamma\Big(\frac{t_1}{2} + \frac{\nu_1 - \nu_2}{4}\Big)\Gamma\Big(\frac{t_1}{2} - \frac{\nu_1 - \nu_2}{4}\Big)\Gamma\Big(\frac{t_2}{2} + \frac{\nu_1 + \nu_2}{4}\Big)\Gamma\Big(\frac{t_2}{2} - \frac{\nu_1 + \nu_2}{4}\Big)$$
$$\cdot \Gamma\Big(\frac{s_1}{2}\Big)\Gamma\Big(\frac{s_1 - t_1 - t_2}{2}\Big)\Gamma\Big(\frac{s_2 - t_1}{2}\Big)\Gamma\Big(\frac{s_2 - t_2}{2}\Big).$$

3. Spinor L-function

We recall Novodvorsky's zeta integral for spinor L-function ([N], see also [Ic]). For a generic cusp form $\varphi \in \Pi$ and $s \in \mathbf{C}$, Novodvorsky considered the integral

$$Z(s, \varphi) = \int_{\mathbf{Q}^\times \backslash \mathbf{A}^\times} \int_{(\mathbf{Q}\backslash\mathbf{A})^3} \varphi(\left(\begin{array}{cc|cc} 1 & x_0 & x_1 & \\ & 1 & & \\ \hline & & 1 & \\ & z & -x_0 & 1 \end{array}\right)\left(\begin{array}{cc|cc} y & & & \\ & y & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right))$$
$$\cdot \psi(x_0) |y|_{\mathbf{A}}^{s-1/2} \, dx_0 dx_1 dz d^\times y,$$

where ψ is a nontrivial additive character of $\mathbf{Q}\backslash\mathbf{A}$. By the cuspidality of φ, this integral converges absolutely for all s and defines an entire function. Then the basic identity and unramified computation tells us that

$$Z(s, \varphi) = \int_{\mathbf{A}^\times} \int_{\mathbf{A}} W_\varphi(\left(\begin{array}{cc|cc} y & & & \\ & y & & \\ \hline & & 1 & \\ & x & & 1 \end{array}\right)) |y|_{\mathbf{A}}^{s-3/2} \, dx d^\times y$$
$$= L_1^S(s, \Pi) \cdot \prod_{v \in S} Z^{(v)}(s, W^{(v)})$$

for $\mathrm{Re}(s) >> 0$. Here S means a finite set of pleces of $\mathbf{Q}$ containing archimedean places, such that Π_v is unramified principal series outside S. $L_1^S(s, \Pi) = \prod_{v \notin S} L_1(s, \Pi_v)$ is the partial spinor L-function. $W_\varphi = \prod_v W^{(v)}$ is the global Whittaker function attached to φ:

$$W_\varphi(g) = \int_{(\mathbf{Q}\backslash\mathbf{A})^4} \varphi(n(x_0, x_1, x_2, x_3)g)\psi(-x_0 - x_3) \, dx_0 dx_1 dx_2 dx_3,$$

and $Z^{(v)}(s, W^{(v)})$ is the local zeta integral. Especially our target is

$$Z^{(\infty)}(s, W^{(\infty)}) = \int_{\mathbf{R}^\times}\int_{\mathbf{R}} W^{(\infty)}(\left(\begin{array}{cc|cc} y & & & \\ & y & & \\ \hline & & 1 & \\ & x & & 1 \end{array}\right))|y|^{s-3/2} dx d^\times y,$$

and we want to find a vector $v \in \Pi_\infty$ to attain the local functional equation:

$$\frac{Z^{(\infty)}(1-s, \widetilde{W_v})}{L_1(1-s, \Pi_\infty^\vee)} = \varepsilon(s, \Pi_\infty, \psi_\infty)\frac{Z^{(\infty)}(s, W_v)}{L_1(s, \Pi_\infty)}.$$

Here $F(g) = \omega_\Pi(\nu(g))^{-1}F(g\left(\begin{array}{cc|cc} & & & 1 \\ & & -1 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right))$ with ω_Π the central character of Π and $\nu(g)$ the similitude factor of $g \in GSp(4, \mathbf{R})$, and $\Pi^\vee = \{\widetilde{F} \mid F \in \Pi\}$ is a contragredient representation of Π.

By using explicit formulas of Whittaker functions on $(G)Sp(4, \mathbf{R})$ given in the previous section, we give some examples of the computation of $Z^{(\infty)}(s, W_v)$ for a suitable vector $v \in \Pi_\infty$. These results are given by Moriyama in case of (1), (2) and by Moriyama and the author in case of (4).

Proposition 3.1. *(1)* [Mo-2] *If* $\Pi_\infty|_{Sp(4,\mathbf{R})} = \pi_{(\lambda_1,\lambda_2)} \oplus \pi_{(-\lambda_2,-\lambda_1)}$ $(1-\lambda_1 \le \lambda_2 \le 0)$*, then*

$$\frac{Z^{(\infty)}(s, W_{v_k})}{L_1(s, \Pi_\infty)} = (i)^{-k}\binom{d}{k}\int_{-i\infty}^{i\infty}\frac{ds_1}{2\pi i}(4\pi)^{-s_1+\lambda_1+1}\frac{\Gamma(s_1 + \frac{-\lambda_1-\lambda_2+1}{2})\Gamma(s_1 + \frac{-\lambda_1+\lambda_2+1}{2})}{\Gamma(\frac{s_1-s-\lambda_1+\lambda_2+k+2}{2})\Gamma(\frac{s_1+s-k+1}{2})}.$$

(2) [Is-Mo] *Let* $\Pi_\infty = I(P_0; \sigma, \nu)$.
(i) If $\sigma = \text{triv.}$*, then*

$$\frac{Z^{(\infty)}(s, W_{v_0})}{L_1(s, \Pi_\infty)} = c\int_{-i\infty}^{i\infty}\frac{ds_1}{2\pi i}\pi^{-2s_1}\frac{\Gamma(s_1 + \frac{\nu_1-\nu_2}{4})\Gamma(s_1 - \frac{\nu_1-\nu_2}{4})\Gamma(s_1 + \frac{\nu_1+\nu_2}{4})\Gamma(s_1 - \frac{\nu_1+\nu_2}{4})}{\Gamma(s_1 + \frac{s+\omega/2}{2})\Gamma(s_1 + \frac{1-(s+\omega/2)}{2})}.$$

(ii) If $\sigma(\gamma_0) = -1$, $\sigma(\gamma_1) = \sigma(\gamma_2) = 1$*, then*

$$\frac{Z^{(\infty)}(s, W_{v_1'})}{L_1(s, \Pi_\infty)} = c'\int_{-i\infty}^{i\infty}\frac{ds_1}{2\pi i}\pi^{-2s_1}\frac{\Gamma(s_1 + \frac{\nu_1-\nu_2}{4})\Gamma(s_1 - \frac{\nu_1-\nu_2}{4})\Gamma(s_1 + \frac{\nu_1+\nu_2}{4})\Gamma(s_1 - \frac{\nu_1+\nu_2}{4})}{\Gamma(s_1 + \frac{s+\omega/2+1}{2})\Gamma(s_1 + \frac{2-(s+\omega/2)}{2})},$$

where the vector $v_1' \in \tau_{(2,0)}$ *is explained in Remark 1 below.*

In our evaluation of zeta integrals Barnes' lemma plays a central role, which says

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(a+s)\Gamma(b+s)\Gamma(c-s)\Gamma(d-s)\,ds = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

Here the path of integration is curved, if necessary, to ensure that the poles of $\Gamma(c-s)\Gamma(d-s)$ lie on the right of the path and $\Gamma(a+s)\Gamma(b+s)$ on the left.

The local functional equation can be confirmed from the integrand of the Mellin-Barnes integral $\int ds_1$.

Remark 1. For $y_1 > 0$, set

$$Z^{(\infty)}(s, y_1, W) := \int_{\mathbf{R}^\times} d^\times y \int_{\mathbf{R}} dx\; W(X(x, y; y_1))|y|^{s-3/2},$$

and

$$Z^{\pm}(s, y_1, W) := \int_0^\infty d^\times y \int_{\mathbf{R}} dx\; W(X(\pm x, \pm y; y_1))y^{s-3/2},$$

with

$$X(x, y; y_1) = \left(\begin{array}{cc|cc} yy_1 & & & \\ & y & & \\ \hline & & y_1^{-1} & \\ & x & & 1 \end{array}\right).$$

Then we have $Z^{(\infty)}(s, W) = Z^{(\infty)}(s, 1, W)$ and

$$\begin{aligned} Z^{(\infty)}(s, y_1, W_v) &= Z^+(s, y_1, W_v) + Z^-(s, y_1, W_v) \\ &= Z^+(s, y_1, W_v) + Z^+(s, y_1, W_{\Pi_\infty(\gamma_0)v}). \end{aligned}$$

This implies that $Z^{(\infty)}(s, y_1, W_v)$ vanishes when $\Pi_\infty(\gamma_0)v = -v$. For example, in the case (2)(ii) in the proposition $Z^{(\infty)}(s, W_{v_0}) = 0$ for the spherical vector $v_0 \in \Pi_\infty$. We set

$$v_1'(g) := v_0(g, X_{(1,1)}), \quad v_1''(g) := v_0(g, X_{(-1,-1)}),$$

where $v(g, X) := \frac{d}{dt}|_{t=0} v(g \exp(tX))$ $(g \in G, X \in \mathrm{Lie}(G))$ and

$$X_{(\pm 1, \pm 1)} = \left(\begin{array}{cc|cc} & 1 & & \pm\sqrt{-1} \\ 1 & & \pm\sqrt{-1} & \\ \hline & \pm\sqrt{-1} & & -1 \\ \pm\sqrt{-1} & & -1 & \end{array}\right).$$

Then $v_1' \in \tau_{(2,0)}$, $v_1'' \in \tau_{(0,-2)}$, and $\Pi_\infty(\gamma_0)v_1' = -v_1''$. We can compute the local zeta integral similarly to the case of class one principal series (2)(i).

Remark 2. (relation between $GSp(4) \times GL(2)$ integrals) Hoffstein and Murty [H-M] obtained the local functional equation for the standard L-function on $GL(3)$ for wave form by explicit computation of local zeta integral. The essence of their idea is to 'reduce to the case of $GL(3) \times GL(2)$.' We give similar interpretation in our case. Consider Mellin transform of $Z^+(s, y_1, W_v)$:

$$\begin{aligned} &\int_0^\infty Z^+(s, y_1, W_v)\, y_1^t d^\times y_1 \\ &\quad = \int_0^\infty d^\times y_1 \int_0^\infty d^\times y\, W_v\big(\mathrm{diag}(\sqrt{y_1}, \sqrt{y}, 1/\sqrt{y_1}, 1/\sqrt{y})\big) y_1^{t/2} y^{s-(t+3)/2} \\ &\qquad \times \int_{\mathbf{R}} dx\, \exp(2\pi\sqrt{-1}\, xy) \cdot (1 + x^2)^{s-(t+3)/2} \Big(\frac{1 - \sqrt{-1}\, x}{\sqrt{1 + x^2}}\Big)^m. \end{aligned}$$

Here an integer m is determined by

$$\Pi_\infty(\left(\begin{array}{cc|cc} 1 & & & \\ & \cos\theta & & \sin\theta \\ \hline & & 1 & \\ & -\sin\theta & & \cos\theta \end{array}\right))v = e^{\sqrt{-1}\, m\theta} v.$$

Then the integral with respect to x becomes Jacquet integral on $GL(2)$, that is principal series Whittaker function on $GL(2)$. Thus the above Mellin transform becomes archimedean part of Novodvorsky's zeta integral on $GSp(2) \times GL(2)$ (see [B1]). Since this integral does not contain 'unipotent integrals,' the computation is relatively easy, and when $m = 0$ Niwa [Ni] computed it:

$$\begin{aligned}
&\int_0^\infty Z_N^+(s, y_1, W_{v_0})\, y_1^t d^\times y_1 \\
&\quad = C\, \pi^{-(s+t+2)} \Gamma\Big(\frac{s}{2} + \frac{\nu_1 - \nu_2}{4}\Big) \Gamma\Big(\frac{s}{2} - \frac{\nu_1 - \nu_2}{4}\Big) \Gamma\Big(\frac{s}{2} + \frac{\nu_1 + \nu_2}{4}\Big) \Gamma\Big(\frac{s}{2} - \frac{\nu_1 + \nu_2}{4}\Big) \\
&\qquad \times \frac{\Gamma(\frac{-s+t+2}{2} + \frac{\nu_1-\nu_2}{4}) \Gamma(\frac{-s+t+2}{2} - \frac{\nu_1-\nu_2}{4}) \Gamma(\frac{-s+t+2}{2} + \frac{\nu_1+\nu_2}{4}) \Gamma(\frac{-s+t+2}{2} - \frac{\nu_1+\nu_2}{4})}{\Gamma(-s + \frac{t+3}{2}) \Gamma(\frac{t+2}{2})}.
\end{aligned}$$

By applying Mellin inversion, we get the desired formula for case (2)(i). See Moriyama [Mo-4] for archimedean theory of L-funtions on $GSp(2) \times GL(2)$ via Novodvorsky's integrals.

At ramified finite places v, Takloo-Bighash [TB] determined local L and ε factors $L_1(s, \Pi_v)$ and $\varepsilon(s, \Pi_v, \psi_v)$. Let $\Lambda_1(s, \Pi) = \prod_v L_1(s, \Pi_v)$ and $\varepsilon_1(s, \Pi) = \prod_v \varepsilon(s, \Pi_v, \psi_v)$ be the completed L-function and ε factor. Our conclusion is as follows:

Theorem 3.2. [Mo-2], [Is-Mo] *Let $\Pi = \otimes_v' \Pi_v$ be a generic cusp form on $GSp(4, \mathbf{A_Q})$ with trivial central character. If Π_∞ is equivalent to (1), (2) or (4) in introduction, the completed spinor L-function $\Lambda_1(s, \Pi)$ can be continued to an entire function of s and satisfies the functional equation*

$$\Lambda_1(s, \Pi) = \varepsilon_1(s, \Pi) \Lambda_1(1 - s, \Pi^\vee).$$

Remark 3. Asgari and Shahidi [A-S] recently obtained a result more general than ours by using Langlands-Shahidi method. In particular they do not impose any restrictions on Π_∞. But we believe that our explicit computation of the local zeta integrals has independent interest.

4. Standard L-function

Except for the doubling method, there are two kinds of integral representations of the standard L-functions for generic cusp forms on $GSp(4)$:

- two variable zeta integral by Bump, Friedberg and Ginzburg [B-F-G],
- Shimura-type zeta integral by Ginzburg, Rallis and Soudry [G-R-S].

4.1. **two variable zeta integral.** For two complex numbers $s_1, s_2 \in \mathbf{C}$ and generic cusp form φ on $GSp(4, \mathbf{A})$, Bump, Friedberg and Ginzburg [B-F-G] considers

$$Z(s_1, s_2, \varphi) = \int_{Z(\mathbf{A})GSp(4,\mathbf{Q})\backslash GSp(4,\mathbf{A})} \varphi(g) E_J(s_1, g) E_S(s_2, g)\, dg$$

Here $Z(\mathbf{A})$ is the center of $GSp(4, \mathbf{A})$ and $E_J(s, g)$ (resp. $E_S(s, g)$) is the Klingen (resp. Siegel) Eisenstein series belonging to the degenerate principal series. Basic identity and unramified computation implies

$$Z(s_1, s_2, \varphi) = L_1^S(s_1, \Pi) L_2^S(s_2, \Pi) \prod_{v \in S} Z^{(v)}(s_1, s_2, W^{(v)}).$$

The local zeta integral at the real place is

$$Z^{(\infty)}(s_1, s_2, W^{(\infty)}) = \int_{Z(\mathbf{R})(N_J \cap N_S) \backslash GSp(4,\mathbf{R})} W^{(\infty)}(g) f_J^{(\infty)}(w_1 g, s_1) f_S^{(\infty)}(w_2 g, s_2)\, dg,$$

where $f_J = \prod_v f_J^{(v)}$ (resp. $f_S = \prod_v f_S^{(v)}$) is the section of Klingen (Siegel) Eisenstein series and w_1 and w_2 are certain elements of the Weyl group of $Sp(4, \mathbf{R})$.

At the present we can carry out the archimedean computation in the following two cases.

Proposition 4.1. *(1) If* $\Pi_\infty = I(P_J; \sigma_{k,+}, \nu)$ *then*

$$Z^{(\infty)}(s_1, s_2, W_{v_0}) = L_1(s_1, \Pi_\infty) L_2(s_2, \Pi_\infty).$$

Here v_0 *is the corner vector in* Π_∞*.*
(2) If $\Pi_\infty = I(P_0; \mathrm{triv}, \nu)$ *then*

$$Z^{(\infty)}(s_1, s_2, W_{v_0}) = L_1(s_1, \Pi_\infty) L_2(s_2, \Pi_\infty).$$

Since we have already proven the functional equations for the spinor L-functions, and thus we get them for the standard L-functions.

4.2. **Shimura type zeta integral.** Recalling the unipotent radical N_J of P_J is isomorphic to Heisenberg group of dimension three, we construct Schrödinger representation w_ψ on $N_J(\mathbf{A}) := \{n(x_0, x_1, x_2, 0) \mid x_0, x_1, x_2 \in \mathbf{A}\}$ and extend it to $N_J(\mathbf{A}) \times \widetilde{SL}(2, \mathbf{A})$. Then we can define a theta function

$$\widetilde{\theta}_\phi(ng) = \sum_{\xi \in \mathbf{Q}} w_\psi(ng)\phi(\xi), \quad (n, g) \in N_J(\mathbf{A}) \times \widetilde{SL}(2, \mathbf{A})$$

with Schwarz-Bruhat function ϕ on $\mathbf{A}$. The zeta integral is

$$Z(s, \varphi) = \int_{SL(2,\mathbf{Q}) \backslash SL(2,\mathbf{A})} \int_{N_J(\mathbf{Q}) \backslash N_J(\mathbf{A})} \varphi|_{N_J \times SL_2}(n, g) \widetilde{E}(g, s) \widetilde{\theta}_\phi(ng)\, dn dg,$$

where $\widetilde{E}(g, s)$ is the metaplectic Eisenstein series on $\widetilde{SL}(2, \mathbf{A})$. The archimedean zeta integral is

$$Z^{(\infty)}(s, W^{(\infty)}) = \int_{N_2(\mathbf{R}) \backslash SL(2,\mathbf{R})} \int_{\mathbf{R}} W^{(\infty)}(\left(\begin{array}{cc|cc} 1 & & & \\ x & 1 & & \\ \hline & & 1 & -x \\ & & & 1 \end{array}\right) \left(\begin{array}{cc|cc} a & & b & \\ & 1 & & \\ \hline c & & d & \\ & & & 1 \end{array}\right))$$
$$\cdot\, w_\psi(n(x, 0, 0, 0)g)\phi(0) \widetilde{f}_s^{(\infty)}(g)\, dx dg,$$

with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$ and $N_2(\mathbf{R}) = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbf{R}\}$. Here is the example of the computation.

Proposition 4.2. *If* $\Pi_\infty = I(P_J; \sigma_{k,+}, \nu)$, *we have*

$$\frac{Z^{(\infty)}(s, W_{v_0})}{L_2(s, \Pi_\infty)} = c \int_{-i\infty}^{i\infty} \frac{ds_1}{2\pi i} (4\pi)^{-s_1} \frac{\Gamma(s_1 - k + \frac{\nu_1}{2})\Gamma(s_1 - k - \frac{\nu_1}{2})\Gamma(s_1 - \frac{k}{2} - 1)}{\Gamma(s_1 - k + \frac{s}{2})\Gamma(s_1 - k + \frac{1-s}{2})}.$$

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