Irreducible representations of $GSp_4(F)$ over a p-adic field F

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Abstract

This is an exposition of the classification of irreducible admissible non-cuspidal representations of the group $GSp_4(F)$, where F is a non-archimedean local field of characteristic zero.

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0 On this note

Let F be a non-archimedean local field, which we assume is of characteristic zero for simplicity. The purpose of this note is to explain the classification of the irreducible admissible representations of the group $GSp_4(F)$. In order to classify the irreducible admissible representations of a p-adic reductive group $G(F)$, one has to solve the following two problems. First one has to describe the irreducible *cuspidal* representations of Levi subgroups. Then it is necessary to describe the irreducible subquotients of the *parabolically induced representations* from those irreducible cuspidal representations. Both problems are difficult and not yet solved in the general case.

For the present group $GSp_4(F)$, only the irreducible cuspidal representations of proper Levi subgroups are known. Hence, in this note, we restrict ourselves to the classification of irreducible *non-cuspidal* representations.

This classification was first extensively studied by Rodier, who classified the irreducible Iwahori spherical representations [Rod88]. Waldspurger [Wal87] and Shahidi [Sha90] made principal contributions to the classification of the irreducible constituents of induced representations from maximal parabolic subgroups. After these results, the classification was completed by Sally-Tadić [ST93]. These works rely (often implicitly) on various techniques in the non-commutative harmonic analysis on p-adic groups developed by Harish-Chandra, Silberger, Bernstein-Zelevinsky, and many others. Our task is to explain most (if not all) of the argument involved in the paper [ST93], but in a suitably concise manner which is accessible for many "users" of the result. On the other hand, the organizer of the workshop suggested us to include the classification for $GL_2(F)$, in order to illustrate some elementary principles. Following these requirements and suggestions, we arrived at the following structure of the exposition.

We first collect some elementary definition and results in the representation theory of p-adic reductive groups in §1. The base of our construction of irreducible representations is the notion of *cuspidal support* (or *infinitesimal character*) of representations introduced by Bernstein-Zelevinsky. The Langlands classification will be used both to compute the reducibility of parabolically induced representations and to describe the irreducible representations.

§2 is devoted to the classification of the irreducible admissible representation of $GL_2(F)$. Since this is an introductory exercise toward the group $GSp_4(F)$, we adopt the argument using the intertwining operator, the Langlands classification and Harish-Chandra's commuting algebra theorem. (As opposed to the one given by Jacquet-Langlands[JL70], which relies heavily on the Whittaker or Kirillov models.) Besides the purpose of illustrating fundamental strategies, the classified irreducible representations also serve as building blocks of representations of $GSp_4(F)$.

Then we turn to the group $GSp_4(F)$ in §3. After preparing some notation, we state our problem in §3.1. We have three conjugacy classes of proper parabolic subgroups B, P_1, P_2 of GSp_4, and the classification of the irreducible subquotients of induced from cuspidal representations exhibits different types of difficulties according to the parabolic subgroups. Since the case of the Borel subgroup B is the longest, we begin our study with this case. Using the Langlands classification and the commuting algebra theorem as well as the functional equation of intertwining operators, we obtain a necessary and sufficient condition for induced representations from B to be reducible.

Using the result of the previous section, the classification is given in §4. The contributions of Sally-Tadić, namely, description of the irreducible constituents of induced modules from B is given in §§4.1, 4.2. We include almost complete proofs, because each case needs its own argument and each such argument is important. In §§4.3, 4.4, the irreducible subquotients of induced from cuspidal representations of maximal parabolic subgroups P_1, P_2 are classified. This part is due to Shahidi (some part by Waldspurger). Here again we give a detailed proof, in order to clarify the role played by Langlands' study of base change lifting for GL_2.

In the final section §5, we give tables of the unitary dual of $GSp_4(F)$. Again the result is due to many authors: Rodier, Shahidi, Sally-Tadić. But the relatively new contribution is the one on the topology of unitary dual developed by Tadić [Tad88].

Finally, an excuse for our notation might be necessary. Throughout the note we adopt the notational convention of J. Arthur, not those of Sally-Tadić. This is because we need to explain every phase of the classification, not only those on Jacquet modules along standard parabolic subgroups.

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1 Basic facts

In this section, we collect basic facts in the representation theory of p-adic reductive groups, which will be used in this note. We often omit proofs and give precise references instead.

1.1 Smooth and admissible representations

First let $\boldsymbol{G}$ be a locally compact totally disconnected group (*i.e.*, an *ℓ-group* in the sense of [BZ76]). In this note, a *smooth representation* of $\boldsymbol{G}$ means a complex representation (π, V) of the abstract group $\boldsymbol{G}$ such that any $v \in V$ is fixed by an open subgroup of $\boldsymbol{G}$. For any representation (π, V) of $\boldsymbol{G}$, its *smooth part*

$$V^\infty := \{v \in V \,|\, \text{the stabilizer of } v \text{ is open}\}$$

is a smooth representation of $\boldsymbol{G}$. We write $\mathcal{R}(\boldsymbol{G})$ for the (abelian) category of smooth representations of $\boldsymbol{G}$. It is obvious that $\mathcal{R}(\boldsymbol{G})$ is closed under taking submodules and quotients. $\mathrm{Irr}(\boldsymbol{G})$ denotes the set of isomorphism classes of irreducible objects in $\mathcal{R}(\boldsymbol{G})$. For $(\pi, V) \in \mathrm{Irr}(\boldsymbol{G})$, Schur's lemma $\mathrm{End}_{\boldsymbol{G}}(\pi) \simeq \mathbb{C}$ holds. In particular, the center $\boldsymbol{Z}$ of $\boldsymbol{G}$ acts by a character $\omega_\pi : \boldsymbol{Z} \to \mathbb{C}^\times$. We call ω_π the *central character* of π.

For $(\pi, V) \in \mathcal{R}(\boldsymbol{G})$, its dual representation π^* on the dual space V^* of V is defined by

$$\langle \pi^*(g)v^*, v\rangle = \langle v^*, \pi(g^{-1})v\rangle, \quad g \in \boldsymbol{G},\, v^* \in V^*,\, v \in V.$$

Its smooth part is denoted by $(\pi^\vee, V^\vee)$ and is called the *contragredient* of (π, V). In general, we have $(V^\vee)^\vee \subset V$ but these two spaces may not coincide. For $v \in V$ and $v^\vee \in V^\vee$, the function

$$f_{v,v^\vee}(g) := \langle \pi(g)v, v^\vee\rangle, \quad g \in \boldsymbol{G}$$

is called a *matrix coefficient* of (π, V).

A smooth representation (π, V) of $\boldsymbol{G}$ is *admissible* if for any open subgroup $K \subset \boldsymbol{G}$, its fixed part $V^K := \{v \in V \,|\, \pi(k)v = v, \forall k \in K\}$ is finite dimensional. Let $\mathcal{R}_{\mathrm{adm}}(\boldsymbol{G})$ be the full subcategory of admissible objects in $\mathcal{R}(\boldsymbol{G})$. $\mathcal{R}_{\mathrm{adm}}(\boldsymbol{G})$ is closed under taking submodules, quotients and duals. That is, the dual of an admissible representation is again admissible. We write $C_c^\infty(\boldsymbol{G})$ for the space of locally constant compactly supported functions on $\boldsymbol{G}$. Also we fix a left invariant measure dg on $\boldsymbol{G}$. For $(\pi, V) \in \mathcal{R}_{\mathrm{adm}}(\boldsymbol{G})$, the operator

$$\pi(f) : V \ni v \longmapsto \int_{\boldsymbol{G}} f(g)\pi(g)v\, dg \in V$$

is well-defined[1] and is of finite rank. Its trace $\mathrm{tr}\pi(f)$ is the *distribution character* of π evaluated at f.

$(\pi, V) \in \mathcal{R}(\boldsymbol{G})$ is said to be *finite* if any matrix coefficient of π is compactly supported. The definition implies that any finitely generated finite representation is admissible [BZ76, 2.41]. Moreover, the irreducible finite representations can be cut out from any smooth representation as follows. We write $\mathrm{Irr}_f(\boldsymbol{G})$ for the set of finite classes in $\mathrm{Irr}(\boldsymbol{G})$.

Theorem 1.1 ([BZ76] 2.44)**.** *(i) Suppose* $\tau \in \mathrm{Irr}_f(\boldsymbol{G})$*. Then any* $(\pi, V) \in \mathcal{R}(\boldsymbol{G})$ *has a decomposition* $(\pi, V) \simeq (\pi[\tau], V[\tau]) \oplus (\pi[\tau]^\perp, V[\tau]^\perp)$*, where* $\pi[\tau]$ *is isomorphic to a direct sum of* τ *and* $\pi[\tau]^\perp$ *has no irreducible subquotient isomorphic to* τ*.*
(ii) For any $(\pi, V) \in \mathcal{R}(\boldsymbol{G})$*,* $V_f := \sum_{\tau \in \mathrm{Irr}_f(\boldsymbol{G})} V[\tau]$ *is a completely reducible finite subrepresentation of* (π, V)*, and* V/V_f *has no finite subquotient.*

1.2 Parabolic induction and Jacquet functor

Now let F be a non-archimedean local field of characteristic zero. We write $|\ |_F$ for its module. $\mathcal{O} \supset \mathfrak{p}$ denotes the maximal compact subring of F and its unique maximal ideal, respectively. For any connected reductive group G defined over F, the group $G(F)$ of its F-rational points is an ℓ-group. As was stated in the title, our general problem is the following.

Problem 1.2. *Describe the set* $\mathrm{Irr}(G(F))$*.*

As in the case of real groups, *parabolic induction* is our principal tool to construct representations of $G(F)$. Let $P = MU \subset G$ be a parabolic subgroup, where M is a Levi component and U the unipotent radical. For $(\pi, V) \in \mathcal{R}(M(F))$, we write $I_P^G(V)$ for the space of smooth (*i.e.*, locally constant) functions $\phi : G(F) \to V$ such that

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[1] Note that the integral is actually a finite sum.

- there exists an open subgroup $K_\phi \subset G(F)$ such that $\phi(gk) = \phi(g)$, $g \in G(F)$, $k \in K_\phi$;
- $\phi(umg) = \delta_P(m)^{1/2}\pi(m)\phi(g)$, $u \in U(F)$, $m \in M(F)$, $g \in G(F)$.

Here, δ_P denotes the modular character of $P(F)$ [2]. The first condition actually follows from the smoothness of ϕ. Then the representation $(I_P^G(\pi), I_P^G(V))$ given by

$$I_P^G(\pi, g)\phi(x) := \phi(xg), \quad g \in G(F),\ \phi \in I_P^G(V)$$

is a smooth representation of $G(F)$. This gives the *parabolic induction* functor $I_P^G : \mathcal{R}(M(F)) \rightsquigarrow \mathcal{R}(G(F))$. One can easily verify the following properties.

(i) I_P^G sends $\mathcal{R}_{\mathrm{adm}}(M(F))$ to $\mathcal{R}_{\mathrm{adm}}(G(F))$.

(ii) I_P^G sends finitely generated $M(F)$-modules to finitely generated $G(F)$-modules.

(iii) If $P_1 = M_1U_1 \supset P = MU$, then $I_{P_1}^G(I_{P\cap M_1}^{M_1}(\pi))$ is naturally isomorphic to $I_P^G(\pi)$.

Next, for $(\pi, V) \in \mathcal{R}(G(F))$, we write

$$\begin{aligned} V(U) &:= \mathrm{span}\{\pi(u)v - v \,|\, u \in U(F),\ v \in V\}, \\ V_P &:= V/V(U) \quad \text{(the } U(F)\text{-coinvariant of } V). \end{aligned}$$

Writing $j_P : V \twoheadrightarrow V_P$ for the natural projection, we define a representation $(\pi_P, V_P) \in \mathcal{R}(M(F))$ by

$$\pi_P(m)j_P(v) := \delta_P(m)^{-1/2}j_P(\pi(m)v), \quad m \in M(F),\ v \in V.$$

This is called the *Jacquet module* of (π, V) along P. The functor $r_P^G : \mathcal{R}(G(F)) \ni (\pi, V) \mapsto (\pi_P, V_P) \in \mathcal{R}(M(F))$ is the *Jacquet functor* along P. It is known (but less obvious) that:

(i) r_P^G sends $\mathcal{R}_{\mathrm{adm}}(G(F))$ to $\mathcal{R}_{\mathrm{adm}}(M(F))$ (Jacquet's lemma, [BZ76, 3.14]).

(ii) r_P^G sends finitely generated representations to finitely generated ones.

One can easily verify the following variant of the Frobenius reciprocity.

Proposition 1.3. *Let $P = MU \subset G$ be a parabolic subgroup and take $\tau \in \mathcal{R}(M(F))$, $\pi \in \mathcal{R}(G(F))$. Then we have a canonical isomorphism*

$$\mathrm{Hom}_{G(F)}(\pi, I_P^G(\tau)) \cong \mathrm{Hom}_{M(F)}(\pi_P, \tau).$$

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[2] That is, for any left invariant measure μ_P on $P(F)$, $\delta_P\mu_P$ is right $P(F)$-invariant.

Bruhat filtration Also we need the following result on the composite of Jacquet and induction functors. Fix a maximal F-split torus A_0 in G. This amounts to a choice of a minimal Levi subgroup $M_0 := \mathrm{Cent}(A_0, G) \subset G$. We write $\mathcal{P}_0 = \mathcal{P}(M_0)$ for the set of (minimal) parabolic subgroups of G having M_0 as a Levi component. The (relative) Weyl group $\mathrm{Norm}(A_0, G)/M_0$ of A_0 in G is denoted by $W^G = W$. We fix a system of representatives $\{\widetilde{w} \,|\, w \in W\}$ of W in $\mathrm{Norm}(A_0, G(F))$. For any $P_0, P_0' \in \mathcal{P}_0$, we have the (relative) *Bruhat decomposition* [BT65, 5.15]

$$G(F) = \coprod_{w \in W} P_0(F)\widetilde{w}P_0'(F).$$

Next let $\mathcal{F}_0 = \mathcal{F}(M_0)$ be the set of parabolic subgroups of G containing M_0. Each $P \in \mathcal{F}_0$ has a unique Levi component M containing M_0. We always take this Levi decomposition for $P \in \mathcal{F}_0$. For $P = MU$, $P' = M'U' \in \mathcal{F}_0$, we fix a system of representatives ${}_{P'}W_P$ of $W^{M'}\backslash W/W^M$. Then the above decomposition yields

$$G(F) = \coprod_{w \in {}_{P'}W_P} P'(F)\widetilde{w}P(F). \tag{1.1}$$

We fix a total order $\geq$ on ${}_{P'}W_P$ such that

- $G(F)_{\geq w} := \coprod_{\substack{w \in {}_{P'}W_P \\ w_1 \geq w}} P(F)\widetilde{w}_1^{-1}P'(F)$ is open in $G(F)$;
- $G(F)_{> w} := \coprod_{\substack{w \in {}_{P'}W_P \\ w_1 > w}} P(F)\widetilde{w}_1^{-1}P'(F)$ is open in $G(F)_{\geq w}$.

Here, as usual, $w_1 > w$ means $w_1 \geq w$ and $w_1 \neq w$. We remark that any parabolic subgroup of G is conjugate to some element of $\mathcal{F}_0$ [BT65, 5.14].

Now take $(\pi, V) \in \mathcal{R}(M(F))$. For each $w \in {}_{P'}W_P$, we set

$$\mathscr{F}^w := \{\phi \in I_P^G(V) \,|\, \mathrm{supp}\,\phi \subset G(F)_{\geq w}\}, \quad \mathscr{F}^{>w} := \sum_{\substack{w' \in {}_{P'}W_P \\ w' > w}} \mathscr{F}^{w'}.$$

$\{\mathscr{F}^w\}_{w \in {}_{P'}W_P}$ is a decreasing filtration of $I_P^G(\pi)|_{P'(F)}$, so that we can consider the Jacquet modules $\mathscr{F}^w_{P'} \in \mathcal{R}(M'(F))$, $(w \in {}_{P'}W_P)$. For any subgroup $H \subset G$ containing M_0, we write $w(H) := \mathrm{Ad}(\widetilde{w})H$.

Theorem 1.4 ([Wal03] I.2.5)**.** *In the above notation, $\mathscr{G}^w_{P'} := \mathscr{F}^w_{P'}/\mathscr{F}^{>w}_{P'}$ is isomorphic to $I^{M'}_{w(P)\cap M'}(w(\pi_{w^{-1}(P')\cap M}))$. Note that the isomorphism class of $w(\pi_{w^{-1}(P')\cap M}) := \pi_{w^{-1}(P')\cap M} \circ \mathrm{Ad}(\widetilde{w})^{-1}$ is independent of the choice of the representative $\widetilde{w}$ for w.*

1.3 Harish-Chandra's theorem

Let A_G be the maximal F-split torus in the center Z of G. $X^*(G)_F := \mathrm{Hom}_{F\text{-grp}}(G, \mathbb{G}_m)$ denotes the group of F-rational characters of G. We write $\mathfrak{a}_G := \mathrm{Hom}(X^*(G)_F, \mathbb{R})$ and define the homomorphism $H_G : G(F) \to \mathfrak{a}_G$ by

$$\langle \chi, H_G(g) \rangle = \log|\chi(g)|_F, \quad \chi \in X^*(G)_F,\ g \in G(F).$$

Its kernel $G(F)^1 = \bigcap_{\chi \in X^*(G)_F} \ker|\chi|_F$ has a compact center and $G(F)^1 A_G(F)$ is of finite index in $G(F)$.

Irreducible finite representations of Levi subgroups of $G(F)$ should serve as building blocks for the category $\mathcal{R}(G(F))$. But, since any matrix coefficient $f_{v,v^\vee}$ of $\pi \in \mathrm{Irr}(G(F))$ satisfies

$$f_{v,v^\vee}(zg) = \omega_\pi(z) f_{v,v^\vee}(g), \quad z \in Z(F),\ g \in G(F),$$

$G(F)$ has no finite representations if $Z(F)$ is not compact (*i.e.*, $A_G \neq \{1\}$). So we introduce the following alternative notion of finite representations called cuspidal representations. We say $(\pi, V) \in \mathcal{R}(G(F))$ is (*super*) *cuspidal* if its restriction to $G(F)^1$ is finite. Applying the Mackey theory to $G(F)^1 A_G(F) \subset G(F)$, one can verify that any irreducible cuspidal representation is admissible. We write $\mathrm{Irr}_0(G(F))$ for the set of cuspidal classes in $\mathrm{Irr}(G(F))$.

Theorem 1.5 (Harish-Chandra, [BZ76] 3.21)**.** *$(\pi, V) \in \mathcal{R}(G(F))$ is cuspidal if and only if $\pi_P = 0$ for any proper parabolic subgroup $P \subsetneq G$.*

This theorem has the following consequences.

Corollary 1.6. *(i) Any $\pi \in \mathrm{Irr}(G(F))$ is isomorphic to a submodule of $I_P^G(\sigma)$ for some $P = MU \in \mathcal{F}_0$ and $\sigma \in \mathrm{Irr}_0(M(F))$.*
(ii) Any $\pi \in \mathrm{Irr}(G(F))$ is admissible.

Proof. (i) Take $P = MU \in \mathcal{F}_0$ which is minimal among those satisfying $\pi_P \neq 0$. Th.1.5 implies that π_P is a finitely generated cuspidal representation of $M(F)$. By virtue of Zorn's lemma, one can take an irreducible quotient σ of π_P. Now the Frobenius reciprocity (Prop.1.3) shows

$$\mathrm{Hom}_{G(F)}(\pi, I_P^G(\sigma)) \cong \mathrm{Hom}_{M(F)}(\pi_P, \sigma) \neq 0.$$

(ii) In (i), the submodule π of an admissible module $I_P^G(\sigma)$ is certainly admissible. $\square$

1.4 Cuspidal support

Fix a minimal parabolic subgroup $P_0 = M_0 U_0 \in \mathcal{F}_0$ and write $\mathcal{F}(P_0)$ for the set of parabolic subgroups of G containing P_0. An element of $\mathcal{F}(P_0)$ is called a *standard* parabolic subgroup. $\mathcal{F}(P_0)$ is a system of representatives of the $G(F)$-conjugacy classes of parabolic subgroups [BT65, 5.14]. $P = MU, P' = M'U' \in \mathcal{F}_0$ are said to be *associated* if M and M' are conjugate: $M' = w(M)$, $\exists w \in W$. For $P \in \mathcal{F}(P_0)$, we write $[P]_{P_0}$ for the set of $P' \in \mathcal{F}(P_0)$ associated with P. Also let $W(M) := \{w \in W \mid w(M) = M\}/W^M$ be the Weyl group of M in G.

Proposition 1.7. *Take $P = MU \in \mathcal{F}_0$ and $\sigma \in \mathrm{Irr}_0(M(F))$.*
(i) If $P \subsetneq G$, then $I_P^G(\sigma)$ has no cuspidal subquotient.
(ii) Take a subquotient π of $I_P^G(\sigma)$. If $P' = M'U'$ is a minimal element of $\{P_1 \in \mathcal{F}_0 \mid \pi_{P_1} \neq 0\}$ and σ' is any irreducible subquotient of $\pi_{P'}$, (M', σ') is $G(F)$-conjugate to (M, σ).
(iii) The length of $I_P^G(\sigma)$ is at most $|[P]_{P_0}| \cdot |W(M)|$.
(iv) Both I_P^G and r_P^G send representations of finite length to those with the same property.

Proof. (i) First notice that the subquotients of $I_P^G(\sigma)$ share the central character $\omega_\sigma|_{Z(F)}$. Let π be any cuspidal subquotient of $I_P^G(\sigma)$. By Th.1.1 (ii), $\pi|_{G(F)^1Z(F)}$ is a direct summand of the submodule $(I_P^G(\sigma)|_{G(F)^1})_f \subset I_P^G(\sigma)|_{G(F)^1}$. Since $G(F)^1Z(F) \lhd G(F)$ is of finite index, this shows that π is a submodule of $I_P^G(\sigma)$. But the Frobenius reciprocity Prop.1.3 implies that

$$\mathrm{Hom}_{G(F)}(\pi, I_P^G(\sigma)) \simeq \mathrm{Hom}_{M(F)}(\pi_P = 0, \sigma) = 0.$$

(ii) By definition, we have a short exact sequence $0 \to V'' \to V' \to \pi \to 0$ for some submodules $V', V'' \subset I_P^G(\sigma)$, so that

$$0 \longrightarrow V''_{P'} \longrightarrow V'_{P'} \longrightarrow \pi_{P'} \longrightarrow 0. \tag{1.2}$$

Since σ is cuspidal, each successive quotient of the Bruhat filtration of $I_P^G(\sigma)_{P'} \supset V'_{P'}$ (Th.1.4) simplifies to

$$\mathscr{G}_{P'}^w \simeq \begin{cases} I_{w(P)\cap M'}^{M'}(w(\sigma)) & \text{if } w(M) \subset M', \\ 0 & \text{otherwise.} \end{cases}$$

If P and P' are not associated, this contains no cuspidal subquotient by (i). But since $\pi_{P'}$ is cuspidal by the choice of P', (1.2) forces $\pi_{P'} = 0$. This is a contradiction. Hence P and P' are associated. Then the Bruhat filtration further simplifies to

$$I_P^G(\sigma)_{P'} = \sum_{\substack{w \in {}_{P'}W_P \\ w(M)=M'}} w(\sigma).$$

Since any irreducible subquotient of $\pi_{P'}$ is a subquotient of this module, the assertion follows.

(iii) The union $\{w(\sigma) \,|\, w \in {}_{P'}W_P,\, w(M) = M',\, P' \in [P]_{P_0}\}$ of the sets of irreducible subquotients of $I_P^G(\sigma)_{P'}$, $P' \in [P]_{P_0}$ has the cardinality $|[P]_{P_0}| \cdot |W(M)|$. On the other hand, this is a disjoint union

$$\coprod_{\pi \in \mathrm{JH}(I_P^G(\sigma))} \bigcup_{P' \in [P]_{P_0}} \mathrm{JH}(\pi_{P'})$$

where $\mathrm{JH}(\Pi)$ denotes the multiset of irreducible subquotients of Π. Since we know from (ii) that $\bigcup_{P' \in [P]_{P_0}} \mathrm{JH}(\pi_{P'}) \neq \varnothing$ for any irreducible subquotient π of $I_P^G(\sigma)$, this implies the estimation in the proposition.

(iv) Since any $\pi \in \mathrm{Irr}(M(F))$ is embedded into some $I_{P_1\cap M}^M(\sigma)$ with $\sigma \in \mathrm{Irr}_0(M_1(F))$, $I_P^G(\pi) \subset I_{P_1}^G(\sigma)$ is of finite length by (iii). This proves the assertion for I_P^G. Similarly, any $\pi \in \mathrm{Irr}(G(F))$ is a submodule of some $I_{P_1}^G(\sigma)$ with $\sigma \in \mathrm{Irr}_0(M_1(F))$. Th.1.4 asserts that the length of $\pi_P \subset I_{P_1}^G(\sigma)_P$ is majorized by the sum of the lengths of $I_{w(P_1)\cap M}^M(w(\sigma))$, ($w \in {}_PW_{P_1}$, $w(M_1) \subset M$). These latter modules are of finite length by (iii), and the assertion for r_P^G is proved. □

Let $K\mathcal{R}(G(F))$ be the Grothendieck group of the full subcategory of $\mathcal{R}(\boldsymbol{G})$ consisting of the representations of finite length. The proposition implies that I_P^G and r_P^G define well-defined homomorphisms

$$i_P^G : K\mathcal{R}(M(F)) \longrightarrow K\mathcal{R}(G(F)), \quad r_P^G : K\mathcal{R}(G(F)) \longrightarrow K\mathcal{R}(M(F)).$$

Proposition 1.8. *Let $P = MU$, $P' = M'U' \in \mathcal{F}_0$ and $\sigma \in \mathrm{Irr}_0(M(F))$, $\sigma' \in \mathrm{Irr}_0(M'(F))$.*
(i) $I_P^G(\sigma)$ and $I_{P'}^{G'}(\sigma')$ share a same irreducible subquotient if and only if (M, σ) and (M', σ') are $G(F)$-conjugate.
(ii) In that case, the set of isomorphism classes of irreducible subquotients of $I_P^G(\sigma)$ coincides with that of $I_{P'}^{G'}(\sigma')$.

Proof. The only if part of (i) follows from Prop.1.7 (ii). For (ii) and the if part of (i), see [BZ76, 2.16]. □

For a Levi subgroup $M \subset G$, we write $\mathcal{P}(M)$ for the set of parabolic subgroups of G having M as a Levi component. For any $\pi \in \mathrm{Irr}(G(F))$, Prop.1.8 implies the existence of a pair (M, σ) consisting of a Levi subgroup $M \subset G$ and $\sigma \in \mathrm{Irr}_0(M(F))$, unique up to $G(F)$-conjugacy, such that π is an irreducible subquotient of $I_P^G(\sigma)$ for any $P \in \mathcal{P}(M)$. The conjugacy class $[M, \sigma]$ of (M, σ) is called the *cuspidal support* of π. Now Prob.1.2 is reduced to the following.

Problem 1.9. *Describe the composition series of $I_P^G(\sigma)$ for $P = MU \in \mathcal{F}(P_0)$ and $\sigma \in \mathrm{Irr}_0(M(F))$.*

1.5 The Langlands classification

To analyze the reducibility of induced from cuspidal modules $I_P^G(\sigma)$ as in Prob.1.9, we need the Langlands classification of irreducible admissible representations. We fix once for all a A_0-good maximal compact subgroup $\mathbf{K} \subset G(F)$. Then we have the *Iwasawa decomposition* $G(F) = U(F)M(F)\mathbf{K}$ for any $P = MU \in \mathcal{F}_0$. Using this, we extend $H_M : M(F) \to \mathfrak{a}_M$ to a map $H_P : G(F) \to \mathfrak{a}_M$ by[3]

$$H_P(umk) = H_M(m), \quad u \in U(F), m \in M(F), k \in \mathbf{K}.$$

Discrete series and tempered representations For $(\pi, V) \in \mathcal{R}_{\mathrm{adm}}(G(F))$, we have a weak isotypic decomposition

$$V = \bigoplus_{\chi \in \mathrm{Irr}(A_G(F))} V_\chi, \quad V_\chi := \{v \in V \mid (\pi(a) - \chi(a))^n v = 0, \exists n \in \mathbb{N}\}.$$

We write $\mathcal{E}xp(\pi)$ for the set of $\chi \in \mathrm{Irr}(A_G(F))$ such that $V_\chi \neq 0$ (the set of *exponents* of π).

Each $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^* = \mathfrak{a}_G^* \otimes_{\mathbb{R}} \mathbb{C}$ defines a quasi-character

$$e^\lambda : G(F) \ni g \longmapsto \exp\langle \lambda, H_G(g) \rangle \in \mathbb{C}^\times$$

of $G(F)$. The group $A_{\widehat{G}}$ of quasi-characters e^λ, ($\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$) so obtained is a complex torus of dimension $\dim A_G$. For $(\pi, V) \in \mathcal{R}(G(F))$, we write $(\pi_\lambda := e^\lambda \otimes \pi, V_\lambda := V)$ for its e^λ-twist. If (π, V) admits a central character ω_π, set $\Re\pi : A_G(F) \ni a \mapsto |\omega_\pi(a)| \in \mathbb{R}_+^\times$. We identify this with an element $\Re\pi \in \mathfrak{a}_G^*$ satisfying $|\omega_\pi(a)| = \exp\langle \Re\pi, H_G(a) \rangle$, $a \in A_G(F)$. Thus $\pi \simeq e^{\Re\pi} \otimes \Im\pi$ where $\Im\pi := e^{-\Re\pi} \otimes \pi$ has a unitary central character.

---

[3] A decomposition $g = umk$, $u \in U(F), m \in M(F), k \in \mathbf{K}$ is not unique, but H_P is well-defined.

An admissible representation (π, V) having a unitary central character is *square integrable* if

$$\int_{G(F)/A_G(F)} |f_{v,v^\vee}(g)|^2\, dg$$

is finite for any $v \in V$, $v^\vee \in V^\vee$. We write $\Pi_{\rm disc}(G(F))$ for the set of square integrable classes in $\mathrm{Irr}(G(F))$. It is obvious that the set $\Pi_0(G(F))$ of supercuspidal classes with unitary central characters in $\mathrm{Irr}(G(F))$ is contained in $\Pi_{\rm disc}(G(F))$. An admissible representation having a unitary central character is *tempered* if its distribution character is a tempered distribution on $G(F)$ in the sense of Harish-Chandra. We write $\Pi_{\rm temp}(G(F))$ for the set of tempered classes in $\mathrm{Irr}(G(F))$. It is known that $\pi \in \mathrm{Irr}(G(F))$ is tempered if and only if it is a direct summand of $I_P^G(\delta)$ for some $P = MU \in \mathcal{F}_0$ and $\delta \in \Pi_{\rm disc}(M(F))$ [Wal03, III.4.1].

Finally, an admissible representation (π, V) is *unitarizable* if it admits a $G(F)$-invariant inner product. We write $\Pi(G(F))$ for the set of unitarizable classes in $\mathrm{Irr}(G(F))$. We have the following inclusions:

$$\Pi_0(G(F)) \subset \Pi_{\rm disc}(G(F)) \subset \Pi_{\rm temp}(G(F)) \subset \Pi(G(F)) \subset \mathrm{Irr}(G(F)). \tag{1.3}$$

One can easily strengthen the *Langlands-Casselman criterion* of square integrability [Wal03, III.1.1, 2.2] as follows. A choice of $P = MU \in \mathcal{P}(M)$ specifies the set Σ_P of P-positive roots of A_M. We write $\Sigma_P^{\rm red} \subset \Sigma_P$ for the subset of reduced roots. We also set ${}^+\mathfrak{a}_P^{G,*} := \sum_{\alpha \in \Sigma_P} \mathbb{R}_{>0}\alpha$ and write ${}^+\bar{\mathfrak{a}}_P^{G,*}$ for its closure in $\mathfrak{a}_M^*$.

Proposition 1.10 ([Kon03] Lem.2.4)**.** *Suppose $\pi \in \mathrm{Irr}(G(F))$ has a unitary central character. Let $[M, \sigma]$ be its cuspidal support.*
(i) π is square integrable if and only if $\Re\mathcal{E}xp(\pi_P) \subset {}^+\mathfrak{a}_P^{G,*}$ for any $P \in \mathcal{P}(M)$.
(ii) π is tempered if and only if $\Re\mathcal{E}xp(\pi_P) \subset {}^+\bar{\mathfrak{a}}_P^{G,*}$ for any $P \in \mathcal{P}(M)$.

Intertwining operators Take a Levi subgroup $M \subset G$ containing M_0. Suppose $(\pi, V) \in \mathcal{R}_{\rm adm}(M(F))$ is of finite length. For $P \in \mathcal{P}(M)$, we have a "bundle" of induced representation $I_P^G(\pi_\lambda) \to e^\lambda \in A_{\widehat{M}}$. For $\phi \in I_P^G(V)$, we have a "constant section"

$$e^\lambda \longmapsto \phi_\lambda(g) := e^{\langle \lambda, H_P(g)\rangle}\phi(g) \in I_P^G(V_\lambda)$$

of this bundle.

We define the intertwining integral by

$$\big(J_{P'|P}(\pi_\lambda)\phi\big)(g) := \int_{(U\cap U')(F)\backslash U'(F)} \phi(u'g)\, du', \quad \phi \in I_P^G(V_\lambda).$$

The integral converges absolutely at λ with $\alpha^\vee(\Re\lambda) \gg 0$ for any $\alpha \in \Sigma_P \smallsetminus \Sigma_{P'}$. Here $\alpha^\vee$ denotes the coroot of α ([Kon03, 2.2]). Moreover, $e^\lambda \mapsto J_{P'|P}(\pi_\lambda)\phi_\lambda(g)$ extends to a rational function on the complex torus $A_{\widehat{M}}$. Besides its poles, it defines an *intertwining operator* ($G(F)$-homomorphism) $J_{P'|P}(\pi_\lambda) : I_P^G(\pi_\lambda) \to I_{P'}^G(\pi_\lambda)$ [Wal03, IV.1.1]. If $P_1 = M_1U_1 \in \mathcal{F}$ contains $P, P' \in \mathcal{P}(M)$, we have the following commutative diagram

$$\begin{tikzcd}
I_P^G(\pi_\lambda) \arrow[r, "J_{P'|P}(\pi_\lambda)"] \arrow[d] & I_{P'}^G(\pi_\lambda) \arrow[d] \\
I_{P_1}^G(I_{P\cap M_1}^{M_1}(\pi_\lambda)) \arrow[r, "J_{P'\cap M_1|P\cap M_1}(\pi_\lambda)"] & I_{P_1}^G(I_{P'\cap M_1}^{M_1}(\pi_\lambda))
\end{tikzcd} \tag{1.4}$$

Moreover, if $P, P', P'' \in \mathcal{P}(M)$ satisfy $|\Sigma_{P''}^{\mathrm{red}} \smallsetminus \Sigma_{P'}^{\mathrm{red}}| + |\Sigma_{P'}^{\mathrm{red}} \smallsetminus \Sigma_{P}^{\mathrm{red}}| = |\Sigma_{P''}^{\mathrm{red}} \smallsetminus \Sigma_{P}^{\mathrm{red}}|$, we have the functional equation

$$J_{P''|P'}(\pi_\lambda) \circ J_{P'|P}(\pi_\lambda) = J_{P''|P}(\pi_\lambda) \tag{1.5}$$

for a suitable choice of relevant measures [*loc.cit.* IV.1 (12)].

The Langlands classification Let $P = MU \in \mathcal{F}_0$. We introduce the dual P-positive chamber

$$\mathfrak{a}_P^{*,+} := \{\lambda \in \mathfrak{a}_M^* \,|\, \alpha^\vee(\lambda) > 0,\ \alpha \in \Sigma_P\}.$$

A representation of the form $I_P^G(\tau_\lambda)$, ($\tau \in \Pi_{\mathrm{temp}}(M(F))$, $\lambda \in \mathfrak{a}_P^{*,+}$) is called a *standard module*. We write $\bar{P} = M\bar{U}$ for the parabolic subgroup *opposite* to P with respect to M: $P \cap \bar{P} = M$.

Theorem 1.11 ([Sil78], [Kon03] Th.3.5)**.** *(i) For a standard module $I_P^G(\tau_\lambda)$ as above, $J_P^G(\tau_\lambda) := \mathrm{im} J_{\bar{P}|P}(\tau_\lambda)$ is its unique irreducible quotient.*
(ii) For any $\pi \in \mathrm{Irr}(G(F))$, there exists a standard module $I_P^G(\tau_\lambda)$ such that $\pi \simeq J_P^G(\tau_\lambda)$. Moreover, the W-conjugacy class of (P, τ, λ) is uniquely determined by π.

1.6 Plancherel measure

Take $P = MU, \bar{P} = M\bar{U} \in \mathcal{F}_0$ opposite to each other, and $\delta \in \Pi_{\mathrm{disc}}(M(F))$. There exists a non-empty Zariski open subset $\Omega_\delta \subset A_{\widehat{M}}$ such that $I_P^G(\delta_\lambda)$ is irreducible for any $e^\lambda \in \Omega_\delta$. This together with Schur's lemma implies the existence of a unique rational function $A_{\widehat{M}} \ni e^\lambda \mapsto j(\delta_\lambda) \in \mathbb{C}$ satisfying

$$J_{P|\bar{P}}(\delta_\lambda) \circ J_{\bar{P}|P}(\delta_\lambda) = j(\delta_\lambda)$$

at any $e^\lambda \in \Omega_\delta$. The notation indicates that this is independent of $P \in \mathcal{P}(M)$ [Wal03, IV.3.(1)]. The *Plancherel measure* or *μ-function* of δ_λ is defined by

$$\mu(\delta_\lambda) := \gamma_M \cdot j(\delta_\lambda)^{-1},$$

where γ_M is a certain constant depending only on M and the choice of relevant measures. This $\mu(\delta)$ together with the formal degree of δ form the weight factor of the contribution of $I_P^G(\delta)$ to *Harish-Chandra's Plancherel formula* of $G(F)$ [Wal03, VIII.1.2].

Proposition 1.12 ([Sil79], Cor.5.4.2.3, Lem.5.4.2.4)**.** *Suppose $P = MU \in \mathcal{F}_0$ is maximal and $\sigma \in \Pi_{\mathrm{disc}}(M(F))$.*
(1) If σ is G-regular, *i.e., $W(M)_\sigma := \{w \in W(M) \,|\, w(\sigma) \simeq \sigma\}$ is trivial, then $I_P^G(\sigma_\lambda)$ is irreducible for any $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$.*
(2) Otherwise, $I_P^G(\sigma)$ is reducible if and only if $\mu(\sigma) \neq 0$.
(3) Further suppose $\sigma \in \Pi_0(M(F))$. Then $I_P^G(\sigma_\lambda)$, ($\lambda \in \mathfrak{a}_P^{*,+}$) is reducible if and only if $\mu(\sigma_\nu)$ has a pole at $\nu = \lambda$.

1.7 Harish-Chandra's commuting algebra theorem

We also need Harish-Chandra's description of the endomorphism algebra of an induced representations $I_P^G(\sigma)$, ($P = MU \in \mathcal{F}_0$, $\sigma \in \Pi_{\mathrm{disc}}(M(F))$). This is achieved by aid of certain normalized intertwining operators.

Proposition 1.13 ([Art89] Th.2.1). *Let M and $\sigma \in \Pi_{\mathrm{disc}}(M(F))$ be as above. There exists a family $\{e^\lambda \mapsto r_{P'|P}(\sigma_\lambda)\}_{P,P' \in \mathcal{P}(M)}$ of rational functions on $A_{\widehat{M}}$ satisfying the following properties.*

(i) The normalized intertwining operator $R_{P'|P}(\sigma_\lambda) := r_{P'|P}(\sigma_\lambda)^{-1} J_{P'|P}(\sigma_\lambda)$ *is regular at σ_λ with $\alpha^\vee(\Re\lambda) \geq 0$, $\forall \alpha \in \Sigma_P \setminus \Sigma_{P'}$.*

(ii) The functional equation $R_{P''|P'}(\sigma_\lambda) \circ R_{P'|P}(\sigma_\lambda) = R_{P''|P}(\sigma_\lambda)$ holds for any $P, P', P'' \in \mathcal{P}(M)$ (not only for those satisfying the length condition, cf. (1.5)*).*

(iii) The adjonction formula $(R_{P'|P}(\sigma_\lambda)\phi, \phi') = (\phi, R_{P|P'}(\sigma_\lambda)\phi')$ holds for $\lambda \in i\mathfrak{a}_M^*$, $\phi \in I_P^G(\sigma_\lambda)$, $\phi' \in I_{P'}^G(\sigma_\lambda)$.

(iv) For $P_1 = M_1U_1 \in \mathcal{F}$ and $P, P' \in \mathcal{P}(M)$ contained in P_1, $r_{P'|P}(\sigma_\lambda) = r_{P'\cap M_1|P\cap M_1}(\sigma_\lambda)$.

(v) Suppose G, σ are unramified and $\mathbf{K} \subset G(F)$ is hyperspecial. For a $\mathbf{K}$-invariant vector $\phi \in I_P^G(\sigma)$, $R_{P'|P}(\sigma_\lambda)\phi_\lambda|_{\mathbf{K}}$ is independent of $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$.

For each $w \in W(M)_\sigma$, we set $M_w^+ := M\langle\widetilde{w}\rangle$, so that we have a (usually non-split) extension

$$1 \longrightarrow M \longrightarrow M_w^+ \longrightarrow \langle w \rangle \longrightarrow 1$$

of linear algebraic groups. As $w(\sigma) \simeq \sigma$, any irreducible component σ_w^+ of $\mathrm{ind}_{M(F)}^{M_w^+(F)} \sigma$ is an extension of σ to $M_w^+(F)$. Then

$$A(\sigma_w^+) : I_{w^{-1}(P)}^G(\sigma) \ni \phi(g) \longmapsto \sigma_w^+(\widetilde{w})\phi(\widetilde{w}^{-1}g) \in I_P^G(\sigma)$$

is an intertwiner independent of the representative $\widetilde{w}$. We define

$$R_{P|P}(w,\sigma) : I_P^G(\sigma) \overset{R_{w^{-1}(P)|P}(\sigma)}{\longrightarrow} I_{w^{-1}(P)}^G(\sigma) \overset{A(\sigma_w^+)}{\longrightarrow} I_P^G(\sigma).$$

Theorem 1.14 ([Sil79] Th.5.5.3.2). $\mathrm{End}_{G(F)}(I_P^G(\sigma))$ *is spanned by* $\{R_{P|P}(w,\sigma) \mid w \in W(M)_\sigma\}$.

2 Classification for $GL_2(F)$

To indicate the outline of our argument, we illustrate the classification of the irreducible admissible representations of $GL_2(F)$. For brevity, we write $G_1 := GL_2$ and $B_1 = T_1U_1$ for its Borel subgroup consisting of the upper triangular elements. Set $\mathbf{K}_1 = G_1(\mathcal{O})$, a maximal compact subgroup of $G_1(F)$, so that we have the Iwasawa decomposition $G_1(F) = B_1(F)\mathbf{K}_1$.

Recall that each $\pi \in \mathrm{Irr}(G_1(F))$ can be uniquely written as $\pi = e^{\Re\pi} \otimes \Im\pi$, where $e^{\Re\pi}(g) = |\omega_\pi(\det g)|^{1/2}$ and $\Im\pi$ has a unitary central character. Thus, in what follows, *we classify only the elements of* $\mathrm{Irr}(G_1(F))$ *with unitary central characters.*

2.1 Weil representations of $O(E) \times SL_2(F)$

Take a quadratic extension E/F and write σ for the generator of its Galois group. $\mathrm{N}_{E/F} : E \to F$ denotes the norm. We regard $(E, \mathrm{N}_{E/F})$ as a 2-dimensional quadratic space over F, and write $O(E)$ for its orthogonal group:

$$O(E) = \{g \in \mathrm{End}_F(E) \,|\, \mathrm{N}_{E/F}(gz) = \mathrm{N}_{E/F}(z)\} = \ker \mathrm{N}_{E/F} \rtimes \mathrm{Gal}(E/F).$$

We also consider the "split" case $E = F^2$, $\sigma(x, y) = (y, x)$. Then the orthogonal group for $(E, \mathrm{N}_{E/F})$ is given by

$$O(F^2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in F^\times \right\} \rtimes \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Fixing a non-trivial character $\psi : F \to \mathbb{C}^\times$ of the additive group F, we have the Weil representation $\omega_E = \omega_{\psi,E}$ of $O(E) \times SL_2(F)$. It is a smooth representation and is realized on the space $\mathcal{S}(E)$ of Schwartz-Bruhat functions on E (the *Schrödinger model*). This is characterized by the following explicit formulae [Kud94, § 5]:

$$\omega_E(g)\Phi(v) = \Phi(g^{-1}.v), \quad g \in O(E), \tag{2.1}$$

$$\omega_E\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)\Phi(v) = \omega_{E/F}(a)|a|_F \Phi(v.a), \quad a \in F^\times, \tag{2.2}$$

$$\omega_E\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\Phi(v) = \psi(b\mathrm{N}_{E/F}(v))\Phi(v), \quad b \in F, \tag{2.3}$$

$$\omega_E\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\Phi(v) = \frac{1}{\lambda(E/F, \psi)} \int_E \Phi(v')\psi_E(-v\sigma(v'))\, dv'. \tag{2.4}$$

Here we write $\omega_{E/F} : F^\times/\mathrm{N}_{E/F}(E^\times) \overset{\sim}{\to} \{\pm 1\}$ for the quadratic character of $F^\times$ associated to E/F by the local classfield theory. $\lambda(E/F, \psi) = \gamma_\psi(\mathrm{N}_{E/F})$ denotes *Langlands' λ-factor* for E/F with respect to ψ. Finally, $\psi_E := \psi \circ \mathrm{Tr}_{E/F}$ and the Haar measure in (2.4) is the self-dual one with respect to the duality $(v, v') \mapsto \psi_E(v\sigma(v'))$.

2.2 Classification of $\mathrm{Irr}(G_1(F)) \smallsetminus \mathrm{Irr}_0(G_1(F))$

In order to classify the irreducible admissible representations of $G_1(F)$ which are not cuspidal, it suffices to classify the irreducible constituents of $I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)$ where (χ_1, χ_2) runs over $\mathrm{Irr}(F^\times)^2$ up to the transposition. Since each $\chi \in \mathrm{Irr}(F^\times)$ is uniquely written as $\chi_1[\lambda] := \chi_1|\ |_F^\lambda$, ($\chi_1 = \Im\chi \in \Pi(F^\times)$, $\lambda = \Re\chi \in \mathbb{R}$), our problem in this subsection can be stated as follows.

Problem 2.1. *Classify the irreducible subquotients of the parabolically induced representation $I_{B_1}^{G_1}(\boldsymbol{\chi}_s)$ from*

$$\boldsymbol{\chi}_s = (\chi_1 \otimes \chi_2)_s : T_1(F) \ni \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \longmapsto \chi_1(a)|a|_F^{s/2}\chi_2(d)|d|_F^{-s/2} \in \mathbb{C}^\times,$$

where $\chi_i \in \Pi(F^\times)$ and $s \geq 0$.

The first step toward Prob.2.1 is to determine the set of $t \geq 0$ for which $I_{B_1}^{G_1}(\boldsymbol{\chi}_t)$ are reducible. The Weyl group W^{G_1} of T_1 in G_1 is generated by the image w of $\widetilde{w} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. In place of $J_{\bar{B}_1|B_1}(\boldsymbol{\chi}_s)$, we consider $M(\widetilde{w}, \boldsymbol{\chi}_s) : I_{B_1}^{G_1}(\boldsymbol{\chi}_s) \to I_{B_1}^{G_1}(w(\boldsymbol{\chi}_s)) = I_{B_1}^{G_1}(w(\boldsymbol{\chi})_{-s})$ defined by

$$\big(M(\widetilde{w}, \boldsymbol{\chi}_s)\phi\big)(g) := \big(\widetilde{w} \circ J_{\bar{B}_1|B_1}(\boldsymbol{\chi}_s)\phi\big)(g) = \int_{U_1(F)} \phi(\widetilde{w}^{-1}ug)\, du, \quad \Re s \gg 0. \tag{2.5}$$

In the above, we adopt the invariant measure $du = dx$ on $U_1(F)$, ($u = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$), where dx is the Haar measure on F selfdual with respect to the duality $(x, y) \mapsto \psi(xy)$. The measure on the lower triangular unipotent subgroup $\bar{U}_1(F)$ is chosen in the same way. We analyze $M(\widetilde{w}, \boldsymbol{\chi}_s)$ using the following realization of $I_{B_1}^{G_1}(\boldsymbol{\chi}_s)$.

Principal series as Howe quotients We consider the Weil representation $(\omega_E, \mathcal{S}(E))$ with $E = F^2$. In this case, one can extend it to a smooth representation $(\omega_{F^2}, \mathcal{S}(F^2))$ of $SO(F^2) \times G_1(F)$ by

$$\omega_{F^2}\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)\Phi\begin{pmatrix} x \\ y \end{pmatrix} = \Phi\begin{pmatrix} ax \\ y \end{pmatrix}, \quad a \in F^\times.$$

Taking the partial Fourier transform

$$\widetilde{\Phi}(x, y) := \int_F \Phi\begin{pmatrix} x \\ y' \end{pmatrix} \psi(yy')\, dy', \quad \Phi \in \mathcal{S}(F^2),$$

we pass to the so-called *mixed model* of ω_{F^2}. One can easily verify the explicit formulae

$$\begin{aligned} \omega_{F^2}\Big(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\Big)\widetilde{\Phi}(x, y) &= |t|_F^{-1}\widetilde{\Phi}(t^{-1}(x, y)), \quad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in SO(F^2), \\ \omega_{F^2}\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\Big)\widetilde{\Phi}(x, y) &= \int_{F^2} \widetilde{\Phi}(x', y')\psi(\langle (x', y'), (x, y)\rangle)\, dx'\, dy', \\ \omega_{F^2}(g)\widetilde{\Phi}(x, y) &= \widetilde{\Phi}((x, y)g), \quad g \in G_1(F). \end{aligned} \tag{2.6}$$

Here $\langle w, w' \rangle := w \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) {}^t w'$.

Let $\boldsymbol{\chi} = (\chi_1, \chi_2)$ as above and $s \in \mathbb{C}$. We have the local Hecke map $(\omega_{F^2}, \mathcal{S}(F^2)) \ni \Phi \mapsto \phi_{\Phi, \boldsymbol{\chi}}(s) \in I_{B_1}^{G_1}(\boldsymbol{\chi}_s)$, which is defined by, for $\Re s \gg 0$,

$$\begin{aligned} \phi_{\Phi, \boldsymbol{\chi}}(s, g) &= \frac{\chi_1(\det g)|\det g|_F^{(s+1)/2}}{L(s+1, \chi_1\chi_2^{-1})} \int_{F^\times} \omega_{F^2}\Big(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}, g\Big)\widetilde{\Phi}(0, 1)\chi_1\chi_2^{-1}(t)|t|_F^s\, dt^\times \\ &= \frac{\chi_1(\det g)|\det g|_F^{(s+1)/2}}{L(s+1, \chi_1\chi_2^{-1})} \int_{F^\times} \widetilde{\Phi}((0, t)g)\chi_1\chi_2^{-1}(t)|t|_F^{s+1}\, dt^\times \end{aligned}$$

with $dt^\times = dt/|t|_F$. Note that the integral on the right hand side is a Zeta integral for the Hecke L-factor $L(s+1, \chi_1\chi_2^{-1})$. This converges absolutely for $\Re s \gg 0$ and defines a rational function in q_F^{-s}. Here q_F is the cardinality of the residue field $\mathcal{O}/\mathfrak{p}$ of F. By definition, $L(s+1, \chi_1\chi_2^{-1})$ is the “GCD” of the rational functions so-obtained, and hence $\phi_{\Phi, \boldsymbol{\chi}}(s, g)$ extends to a regular function in q_F^{-s} for any $g \in G_1(F)$.

Lemma 2.2. *$(\omega_{F^2}, \mathcal{S}(F^2)) \ni \Phi \mapsto \phi_{\Phi,\boldsymbol{\chi}}(s) \in I_{B_1}^{G_1}(\boldsymbol{\chi}_s)$ is well-defined and $G_1(F)$-equivariant. Moreover, it is surjective for $\Re s > -1$.*

Proof. One can easily verify the first assertion. As for the last, we first note that the defining integral of $\phi_{\Phi,\chi}(s)$ converges absolutely for $\Re s > -1$. Assume $\Re s > -1$ and take $\phi \in I_{B_1}^{G_1}(\boldsymbol{\chi}_s)$. We can find $\Phi \in \mathcal{S}(F^2)$ such that

$$\widetilde{\Phi}(x,y) = \begin{cases} \dfrac{L(s+1, \chi_1\chi_2^{-1})}{\chi_1(\det k)\operatorname{meas}\mathcal{O}^\times}\phi(k) & \text{if } (x,y) = (0,1)k \text{ for some } k \in \mathbf{K}_1, \\ 0 & \text{if } (x,y) \notin (0,1)\mathbf{K}_1. \end{cases}$$

Here $\operatorname{meas}\mathcal{O}^\times$ denotes the measure of $\mathcal{O}^\times$ with respect to dx. Since the stabilizer of $(0,1)$ in $\mathbf{K}_1$ consists of the elements $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, ($x \in \mathcal{O}$), this $\widetilde{\Phi}$ is well-defined. It is obvious that $\widetilde{\Phi}$ is locally constant and compactly supported. Moreover, we have

$$\begin{aligned} \phi_{\Phi,\boldsymbol{\chi}}(s,k) =& \frac{1}{\operatorname{meas}\mathcal{O}^\times}\int_{F^\times} \widetilde{\Phi}((0,t)k)\chi_1\chi_2^{-1}(t)|t|_F^{s+1}\,dt^\times \\ =& \frac{1}{\operatorname{meas}\mathcal{O}^\times}\int_{F^\times} \widetilde{\Phi}\Big((0,1)\begin{pmatrix} t^{-1} & \\ & t \end{pmatrix} k\Big)\chi_1\chi_2^{-1}(t)|t|_F^{s}\,dt \\ =& \frac{1}{\operatorname{meas}\mathcal{O}^\times}\int_{\mathcal{O}^\times} \phi(k)\,dt = \phi(k) \end{aligned}$$

as desired. ☐

Calculation of the intertwining operator Now we compute the operator $M(\widetilde{w}, \boldsymbol{\chi}_s)$ applied to these $\phi_{\Phi,\boldsymbol{\chi}}(s)$. Thanks to the relation

$$M(\widetilde{w}, \boldsymbol{\chi}_s)\phi\Big(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\Big) = \chi_2(a)\chi_1(d)\left|\frac{a}{d}\right|_F^{(1-s)/2} M(\widetilde{w}, \boldsymbol{\chi}_s)\phi(g),$$

it suffices to calculate $M(\widetilde{w}, \boldsymbol{\chi}_s)\phi_{\Phi,\boldsymbol{\chi}}(s)$ evaluated at $g \in SL_2(F)$. (Note $G_1(F) = B_1(F)SL_2(F)$!) Assuming $\Re s > 0$, both the defining integral of $M(\widetilde{w}, \boldsymbol{\chi}_s)$ and $\phi_{\Phi,\boldsymbol{\chi}}(s)$ are absolutely convergent, so that one can proceed as

$$\begin{aligned} &M(\widetilde{w}, \boldsymbol{\chi}_s)\phi_{\Phi,\boldsymbol{\chi}}(s,g) \\ &\quad= \frac{\chi_1(\det g)|\det g|_F^{(s+1)/2}}{L(s+1, \chi_1\chi_2^{-1})}\int_{U_1(F)}\int_{F^\times} \widetilde{\Phi}((0,t)\widetilde{w}^{-1}ug)\chi_1\chi_2^{-1}(t)|t|_F^{s+1}\,dt^\times\,du \\ &\quad= \frac{\chi_1(\det g)|\det g|_F^{(s+1)/2}}{L(s+1, \chi_1\chi_2^{-1})}\int_{F^\times}\int_{F} \widetilde{\Phi}((t,tx)g)\,dx\,\chi_1\chi_2^{-1}(t)|t|_F^{s+1}\,dt^\times \\ &\quad= \frac{\chi_1(\det g)|\det g|_F^{(s+1)/2}}{L(s+1, \chi_1\chi_2^{-1})}\int_{F^\times}\int_{F} \omega_{F^2}(g)\widetilde{\Phi}(t,b)\,db\,\chi_1\chi_2^{-1}(t)|t|_F^{s}\,dt^\times. \end{aligned}$$

It follows from (2.6) and the Fourier inversion formula that

$$\begin{aligned}\int_F \omega_{F^2}\Big(\begin{pmatrix}0&1\\1&0\end{pmatrix}\Big)\widetilde{\Psi}(0,y)\psi(-ty)\,dy &= \int_F\int_{F^2}\widetilde{\Psi}(a,b)\psi(ay)\,da\,db\,\psi(-ty)\,dy\\ &= \int_F\int_F\int_F\widetilde{\Psi}(a,b)\psi(ay)\,da\,\psi(-ty)\,dy\,db\\ &= \int_F\widetilde{\Psi}(t,b)\,db\end{aligned}$$

Putting this into the above equality, we obtain, writing $\widetilde{\Phi}^\wedge := \omega_{F^2}\Big(\begin{pmatrix}0&1\\1&0\end{pmatrix}\Big)\widetilde{\Phi}$,

$$\begin{aligned}&M(\widetilde{w},\boldsymbol{\chi}_s)\phi_{\Phi,\boldsymbol{\chi}}(s,g)\\ &\quad=\frac{\chi_1(\det g)|\det g|_F^{(s+1)/2}}{L(s+1,\chi_1\chi_2^{-1})}\int_{F^\times}\Big(\int_F\omega_{F^2}(g)\widetilde{\Phi}^\wedge(0,y)\psi(-ty)\,dy\Big)\chi_1\chi_2^{-1}(t)|t|_F^s\,dt^\times\end{aligned}$$

writing $f(t)$ for the inner integral,

$$=\frac{\chi_1(\det g)|\det g|_F^{(s+1)/2}}{L(s+1,\chi_1\chi_2^{-1})}Z(f,\chi_1\chi_2^{-1},s).$$

Here,

$$Z(f,\chi,s) := \int_{F^\times} f(t)\chi(t)|t|_F^s\,dt^\times$$

is Tate's zeta integral [Tat79, (3.2)]. If we use the local functional equation

$$\frac{Z(\hat{f},\chi^{-1},1-s)}{L(1-s,\chi^{-1})} = \varepsilon(s,\chi,\psi)\frac{Z(f,\chi,s)}{L(s,\chi)}$$

of the Hecke L-function, the above becomes

$$\begin{aligned}M(\widetilde{w},\boldsymbol{\chi}_s)\phi_{\Phi,\boldsymbol{\chi}}(s,g) &= \frac{\chi_1(\det g)|\det g|_F^{(s+1)/2}}{L(s+1,\chi_1\chi_2^{-1})}\frac{L(s,\chi_1\chi_2^{-1})}{\varepsilon(s,\chi_1\chi_2^{-1},\psi)L(1-s,\chi_1^{-1}\chi_2)}\\ &\quad\times\int_{F^\times}\hat{f}(t)\chi_1^{-1}\chi_2(t)|t|_F^{1-s}\,dt^\times\end{aligned}$$

using the definition of $f(t)$,

$$\begin{aligned}&=\frac{L(s,\chi_1\chi_2^{-1})}{L(s+1,\chi_1\chi_2^{-1})\varepsilon(s,\chi_1\chi_2^{-1},\psi)}\chi_1\chi_2^{-1}(\det g)|\det g|_F^s\\ &\quad\times\chi_2(\det g)|\det g|_F^{(1-s)/2}\int_{F^\times}\widetilde{\Phi}^\wedge((0,t)g)\chi_1^{-1}\chi_2(t)|t|_F^{1-s}\,dt^\times\\ &=\frac{L(s,\chi_1\chi_2^{-1})}{L(s+1,\chi_1\chi_2^{-1})\varepsilon(s,\chi_1\chi_2^{-1},\psi)}\chi_1\chi_2^{-1}(\det g)|\det g|_F^s\phi_{\Phi^\wedge,w(\boldsymbol{\chi})}(-s,g),\end{aligned}$$

provided that $0 < \Re s < 1$. Thus we have the equality

$$M(\widetilde{w}, \boldsymbol{\chi}_s)\phi_{\Phi,\boldsymbol{\chi}}(s, g) = r(w, \boldsymbol{\chi}_s, \psi)\chi_1\chi_2^{-2}(\det g)|\det g|_F^s\phi_{\Phi^\wedge,w(\boldsymbol{\chi})}(-s, g) \tag{2.7}$$

of rational functions in q_F^{-s} (for any $s \in \mathbb{C}$). Here $\Phi^\wedge \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \omega_{F^2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\Phi\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \Phi\left(\begin{smallmatrix} y \\ x \end{smallmatrix}\right)$ and

$$r(w, \boldsymbol{\chi}_s, \psi) := \frac{L(s, \chi_1\chi_2^{-1})}{L(s+1, \chi_1\chi_2^{-1})\varepsilon(s, \chi_1\chi_2^{-1}, \psi)} \tag{2.8}$$

is the *Langlands normalization factor* for $M(\widetilde{w}, \boldsymbol{\chi}_s)$.

Classification of the irreducible non-cuspidal representations

Theorem 2.3. *(i)* $I_{B_1}^{G_1}(\boldsymbol{\chi}_s)$*,* ($\boldsymbol{\chi} = \chi_1 \otimes \chi_2 \in \Pi(T_1(F))$*,* $s \in \mathbb{R}_{\geq 0}$*) is reducible if and only if* $\chi_1 = \chi_2$ *and* $s = \pm 1$*.*
(ii) In that case, we have

$$\begin{aligned}
0 \longrightarrow \chi(\det)\delta^{G_1} \longrightarrow I_{B_1}^{G_1}(\chi[1/2] \otimes \chi[-1/2]) \longrightarrow \chi(\det) \longrightarrow 0,\\
0 \longrightarrow \chi(\det) \longrightarrow I_{B_1}^{G_1}(\chi[-1/2] \otimes \chi[1/2]) \longrightarrow \chi(\det)\delta^{G_1} \longrightarrow 0.
\end{aligned}$$

Here, δ^{G_1} *is an element of* $\Pi_{\mathrm{disc}}(G_1(F))$ *called the* Steinberg representation.

Proof. Recall that $J_{\bar{B}_1|B_1}(\boldsymbol{\chi}_s) = \widetilde{w}^{-1} \circ M(\widetilde{w}, \boldsymbol{\chi}_s)$. Also we have

$$\begin{aligned}
&J_{B_1|\bar{B}_1}(\boldsymbol{\chi}_s)\phi_{\Phi',w(\boldsymbol{\chi})}(-s, g)\\
&\qquad= \int_{U_1(F)} \phi_{\Phi',w(\boldsymbol{\chi})}(-s, \widetilde{w}ug)\, du = \int_{U_1(F)} \phi_{\Phi',w(\boldsymbol{\chi})}(-s, -\widetilde{w}^{-1}ug)\, du\\
&\qquad=\chi_1\chi_2(-1)M(\widetilde{w}, w(\boldsymbol{\chi}_s))\phi_{\Phi',w(\boldsymbol{\chi})}(-s, g)
\end{aligned}$$

applying (2.7),

$$=\chi_1\chi_2(-1)r(w, w(\boldsymbol{\chi}_s), \psi)\chi_1^{-1}\chi_2(\det g)|\det g|_F^{-s}\phi_{\Phi'^\wedge,\boldsymbol{\chi}}(s, g)$$

for $\Re s < 0$. This is valid for all $s \in \mathbb{C}$ since both sides are rational functions in q_F^{-s}. These together with (2.7) yields

$$\begin{aligned}
J_{B_1|\bar{B}_1}(\boldsymbol{\chi}_s)\circ J_{\bar{B}_1|B_1}(\boldsymbol{\chi}_s)\phi_{\Phi,\boldsymbol{\chi}}(s) &= M(\widetilde{w}^{-1}, w(\boldsymbol{\chi}_s)) \circ M(\widetilde{w}, \boldsymbol{\chi}_s)\phi_{\Phi,\boldsymbol{\chi}}(s)\\
&=M(\widetilde{w}^{-1}, w(\boldsymbol{\chi}_s))r(w, \boldsymbol{\chi}_s, \psi)\chi_1\chi_2^{-1}(\det)|\det|_F^s\phi_{\Phi^\wedge,w(\boldsymbol{\chi})}(-s)\\
&=\chi_1\chi_2(-1)r(w, w(\boldsymbol{\chi}_s), \psi)r(w, \boldsymbol{\chi}_s, \psi)\phi_{\Phi,\boldsymbol{\chi}}(s)\\
&=r(w, \boldsymbol{\chi}_s, \psi)r(w, w(\boldsymbol{\chi}_s), \bar{\psi})\phi_{\Phi,\boldsymbol{\chi}}(s).
\end{aligned} \tag{2.9}$$

Here, we have used the relation $\varepsilon(s, \chi, \psi^a) = \chi(a)|a|_F^{s-1/2}\varepsilon(s, \chi, \psi)$ for $\psi^a(x) := \psi(ax)$ [Tat79, (3.2.3)]. Granting Lem.2.2, this shows

$$j(\boldsymbol{\chi}_s) = r(w, \boldsymbol{\chi}_s, \psi)r(w, w(\boldsymbol{\chi}_s), \bar{\psi})$$

for $-1 < \Re s < 1$ and hence for all $s \in \mathbb{C}$. This together with (2.8) give

$$\begin{aligned}\mu(\boldsymbol{\chi}_s) =& \gamma_{T_1} r(w, \boldsymbol{\chi}_s, \psi)^{-1} r(w, w(\boldsymbol{\chi}_s), \bar{\psi})^{-1} \\ =& \gamma_{T_1} \frac{L(s+1, \chi_1\chi_2^{-1})\varepsilon(s, \chi_1\chi_2^{-1}, \psi)}{L(s, \chi_1\chi_2^{-1})} \frac{L(1-s, \chi_1^{-1}\chi_2)\varepsilon(-s, \chi_1^{-1}\chi_2, \bar{\psi})}{L(-s, \chi_1^{-1}\chi_2)}.\end{aligned} \tag{2.10}$$

Now we apply Prop.1.12 (1) to this. In order for $I_{B_1}^{G_1}(\boldsymbol{\chi}_s)$ to be reducible, $\boldsymbol{\chi}$ has to be G_1-singular, *i.e.*, $\chi_1 = \chi_2$. Then (2.10) becomes (noting $\varepsilon(s, \chi_1\chi_2^{-1}, \psi)\varepsilon(-s, \chi_1^{-1}\chi_2, \bar{\psi}) = q_F^{\mathrm{ord}\psi}$)

$$\mu(\boldsymbol{\chi}_s) = \gamma_{T_1} q_F^{\mathrm{ord}\psi} \frac{(1-q_F^{-s})(1-q_F^{s})}{(1-q_F^{-s-1})(1-q_F^{s-1})}.$$

This has a double zero at $s = 0$ and a simple pole at $s = 1$. The assertion (i) follows at once from Prop.1.12 (2), (3).

(ii) Since $I_{B_1}^{G_1}((\chi^{\otimes 2})_s) \simeq \chi(\det) I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$, where $\mathbb{1}_{T_1(F)}$ denotes the trivial character of $T_1(F)$, we may assume $\boldsymbol{\chi} = \mathbb{1}_{T_1(F)}$. As in the proof of Prop.1.7, each irreducible constituent of $I_{B_1}^{G_1}(|\ |_F^{1/2} \otimes |\ |_F^{-1/2})$ contributes non-trivially to the cuspidal Jacquet module

$$r_{B_1}^{G_1} \circ i_{B_1}^{G_1}(|\ |_F^{1/2} \otimes |\ |_F^{-1/2}) = |\ |_F^{1/2} \otimes |\ |_F^{-1/2} + |\ |_F^{-1/2} \otimes |\ |_F^{1/2}.$$

We write δ^{G_1} for the constituent such that $(\delta^{G_1})_{B_1} \simeq |\ |_F^{1/2} \otimes |\ |_F^{-1/2}$. Since $\Re\mathcal{E}xp((\delta^{G_1})_{B_1}) = 1 > 0$, δ^{G_1} is square integrable by Prop.1.10. On the other hand, the Langlands quotient $J_{B_1}^{G_1}(|\ |_F^{1/2} \otimes |\ |_F^{-1/2})$ is non-tempered, so that it must be the other constituent of $I_{B_1}^{G_1}(|\ |_F^{1/2} \otimes |\ |_F^{-1/2})$. By Prop.1.8 $I_{B_1}^{G_1}(|\ |_F^{-1/2} \otimes |\ |_F^{1/2})$ share the same irreducible constituents δ^{G_1}, $J_{B_1}^{G_1}(|\ |_F^{1/2} \otimes |\ |_F^{-1/2})$. But

$$I_{B_1}^{G_1}(|\ |_F^{-1/2} \otimes |\ |_F^{1/2}) = \left\{ \phi : G_1(F) \to \mathbb{C} \ \middle|\ \begin{array}{ll} (1) & \phi \text{ is smooth} \\ (2) & \phi(bg) = \phi(g),\ b \in B_1(F), g \in G(F) \end{array} \right\}$$

certainly contains the constant function, *i.e.*, the trivial representation $\mathbb{1}_{G_1(F)}$. Thus $J_{B_1}^{G_1}(|\ |_F^{1/2} \otimes |\ |_F^{-1/2}) \simeq \mathbb{1}_{G_1(F)}$. The theorem is proved. □

2.3 Representations of dihedral type

We fix an algebraic closure $\bar{F}$ of F, so that, from now on, every algebraic extension of F is taken inside it. Write W_F for the *Weil group* of $\bar{F}/F$ [Tat79].

Let E be a quadratic extension of F and write σ for the generator of the Galois group $\mathrm{Gal}(E/F)$. Take $\omega \in \mathrm{Irr}(E^\times)$ and identify this with a quasi-character of the Weil group W_E of E by the local classfield theory (in the sense of Langlands [Mil06, I.8]). We briefly review the construction of $\pi(\omega) \in \mathrm{Irr}(G_1(F))$ from [JL70], which corresponds to the 2-dimensional representation $\mathrm{ind}_{W_E}^{W_F}\omega$ under the local Langlands correspondence [HT01], [Hen00], [Kut80].

Write $G_1(F)_+ := \{g \in G_1(F) \,|\, \det g \in \mathrm{N}_{E/F}(E^\times)\}$ for the moment. Setting $\omega_\circ := \omega|_{SO(E)}$, the $\omega_\circ^{-1}$-isotypic quotient or subspace of the Weil representation $(\omega_E, \mathcal{S}(E))$ is given by

$$\begin{aligned}\mathcal{S}(E)_{\bar{\omega}_\circ} &:= (\mathcal{S}(E) \otimes \mathbb{C}_{\omega_\circ})^{SO_E(F)} \\ &= \{\Phi \in \mathcal{S}(E) \,|\, \Phi(g^{-1}v) = \bar{\omega}_\circ(g)\Phi(v),\ g \in SO_E(F),\ v \in E\}.\end{aligned}$$

Table 1: Table of $\mathrm{Irr}(GL_2(F))$

Representation	Property	Condition	Comment
extraordinary cuspidal	cuspidal	—	appears only if $2\|q_F$
$\pi(\omega)$ dihedral	cuspidal	E/F quadratic ext., $\mathrm{Gal}(E/F) = \langle\sigma\rangle$, $\omega \in \Pi(E^\times)$, $\sigma(\omega) \neq \omega$	$\pi(\omega) \simeq \pi(\omega')$ iff $\mathrm{ind}_{W_E}^{W_F}(\omega) \simeq \mathrm{ind}_{W_{E'}}^{W_F}(\omega')$
$\chi(\det)\delta^{G_1}$ Steinberg	square integrable	$\chi(\det)\delta^{G_1} \hookrightarrow I_{B_1}^{G_1}((\chi^{\otimes 2})_1)$	$\chi(\det)\delta^{G_1} \simeq \chi'(\det)\delta^{G_1}$ iff $\chi = \chi'$
$I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)$ principal series	tempered	$\chi_1, \chi_2 \in \Pi(F^\times)$	$I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)$ $\simeq I_{B_1}^{G_1}(\chi_1' \otimes \chi_2')$ iff $\{\chi_1, \chi_2\} = \{\chi_1', \chi_2'\}$
$I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)$	non-tempered	$\chi_1, \chi_2 \in \mathrm{Irr}(F^\times) \smallsetminus \Pi(F^\times)$ $\chi_1\chi_2^{-1} \neq \|\ \|_F^{\pm 1}$	$I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)$ $\simeq I_{B_1}^{G_1}(\chi_1' \otimes \chi_2')$ iff $\{\chi_1, \chi_2\} = \{\chi_1', \chi_2'\}$
$\chi(\det)$	non-tempered	$\chi \in \mathrm{Irr}(F^\times)$ $I_{B_1}^{G_1}((\chi^{\otimes 2})_1) \twoheadrightarrow \chi(\det)$	1-dimensional

The representation of $SL_2(F)$ on $\mathcal{S}(E)_{\bar{\omega}_\circ}$ is denoted by $\pi(\omega_\circ, \psi_F)$. This is known to be irreducible [Sha04]. One can extend this to $\pi(\omega, \psi) \in \mathrm{Irr}(G_1(F)_+)$ by setting

$$\pi(\omega, \psi)\Big(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\Big)\Phi(v) := \omega(z)|z|_E^{1/2}\Phi(vz), \quad a = \mathrm{N}_{E/F}(z) \in \mathrm{N}_{E/F}(E^\times).$$

Note that the right hand side is independent of $z \in E^\times$ with the norm a. Now we set $\pi(\omega) := \mathrm{ind}_{G_1(F)_+}^{G_1(F)} \pi(\omega, \psi)$.

Theorem 2.4 ([JL70] Th.4.6). *(i) $\pi(\omega)$ is irreducible and independent of ψ.*
(ii) If $\sigma(\omega) := \omega \circ \sigma \neq \omega$, then $\pi(\omega)$ is supercuspidal. Otherwise, ω is of the form $\omega = \chi \circ \mathrm{N}_{E/F}$ for some $\chi \in \mathrm{Irr}(F^\times)$ and $\pi(\omega) \simeq I_{B_1}^{G_1}(\chi \otimes \chi\omega_{E/F})$.

If the residual characteristic of F is odd, any element of $\mathrm{Irr}_0(G_1(F))$ is of the form $\pi(\omega)$ for some (E, ω). Otherwise, we have the so called *extraordinary* cuspidal representations which cannot be obtained in this way. We summarize the classification of the irreducible admissible representations of $GL_2(F)$ obtained above as Table 1.

Remark 2.5. *The proof of the equivalence relation for $\pi(\omega)$ in Table* 1 *requires the cyclic base change lifting [Lan80]. The other equivalences immediately follow from Prop.*1.8.

2.4 The unitary dual $\Pi(G_1(F))$

We review the classification of the unitary dual $\Pi(G_1(F))$ in order to illustrate the construction of the complementary series representations.

Theorem 2.6. *The set $\Pi(G_1(F))$ consists of the following representations.*

Label	*Representation*	*Property*	*Series*
(1)	$\pi \in \Pi_0(G_1(F))$, $\omega_\pi \in \Pi(F^\times)$	*cuspidal*	*discrete series*
(2)	$\chi(\det)\delta^{G_1}$	*square integrable*	*discrete (series ?)*
(3)	$I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)$, $\chi_i \in \Pi(F^\times)^2$	*tempered*	*principal series*
(3)	$I_{B_1}^{G_1}(\chi[s] \otimes \chi[-s])$ $\chi \in \Pi(F^\times)$, $0 < s < 1/2$	*non-tempered*	*complementary series*
(4)	$\chi(\det)$, $\chi \in \Pi(F^\times)$	1-*dimensional*	

Proof. We have only to describe $\Pi(G_1(F)) \smallsetminus \Pi_{\mathrm{temp}}(G_1(F))$, or equivalently, unitarizable representations in the last two rows in Table 1. Clearly $\chi(\det)$ is unitary if and only if so is χ.

Take $\chi_1 \otimes \chi_2 \in \mathrm{Irr}(T_1(F)) \smallsetminus \Pi(T_1(F))$ such that $I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)$ is irreducible. In order for this to be unitarizable, its central character $\chi_1\chi_2$ must be unitary. Thus we may assume $\chi_1 \otimes \chi_2 = \boldsymbol{\chi}_s$ for $\boldsymbol{\chi} = \chi \otimes \chi' \in \Pi(T_1(F))$ and $s \in \mathbb{R}_{>0}$. Next we consider the hermitian pairing $(\cdot,\cdot) : I_{B_1}^{G_1}(\boldsymbol{\chi}_s) \times I_{B_1}^{G_1}(\boldsymbol{\chi}_{-s}) \to \mathbb{C}$ given by

$$\langle \phi, \phi' \rangle := \int_{\mathbf{K}_1} \phi(k)\overline{\phi'(k)}\,dk, \quad \phi \in I_{B_1}^{G_1}(\boldsymbol{\chi}_s),\ \phi' \in I_{B_1}^{G_1}(\boldsymbol{\chi}_{-s}).$$

One can easily verify that this is well-defined and $G_1(F)$-invariant. (Note that the integral can be replaced with that over $B_1(F)\backslash G(F)$.) As this is obviously non-trivial, the irreducibility of $I_{B_1}^{G_1}(\boldsymbol{\chi}_s)$ implies that it is non-degenerate. Thus if $I_{B_1}^{G_1}(\boldsymbol{\chi}_s)$ admits a $G_1(F)$-invariant inner product $(\cdot,\cdot)$, we get an isomorphism $A : I_{B_1}^{G_1}(\boldsymbol{\chi}_s) \xrightarrow{\sim} I_{B_1}^{G_1}(\boldsymbol{\chi}_{-s})$ defined by

$$\langle \phi_1, A(\phi_2) \rangle = (\phi_1, \phi_2), \quad \forall \phi_1 \in I_{B_1}^{G_1}(\boldsymbol{\chi}_s).$$

Granting the equivalence relation in Table1, this forces $\{\chi[s/2], \chi'[-s/2]\} = \{\chi[-s/2], \chi'[s/2]\}$, namely $\chi = \chi'$.

Now, since $I_{B_1}^{G_1}((\chi^{\otimes 2})_s) = \chi(\det) I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$, one may assume $\boldsymbol{\chi} = \mathbb{1}_{T_1(F)}$. It follows from Lem.2.2 and (2.7) that the normalized intertwining operator

$$N(\widetilde{w}, \mathbb{1}_{T_1(F),s}) := r(w, \mathbb{1}_{T_1(F),s}, \psi)^{-1} M(\widetilde{w}, \mathbb{1}_{T_1(F),s}) : I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}) \longrightarrow I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),-s})$$

is holomorphic for $\Re s \geq 0$. Using this, we define a hermitian form $(\cdot,\cdot)_s$ on $I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$ by

$$(\phi, \phi')_s := \int_{\mathbf{K}_1} \phi(k)\overline{N(\widetilde{w}, \mathbb{1}_{T_1(F),s})\phi'(k)}\,dk.$$

It is known that, for any $s \in \mathbb{C}$, there exists $\phi \in I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$ such that $M(\widetilde{w}, \mathbb{1}_{T_1(F),s})\phi \neq 0$ (see [Wal03, p.283]). Since $r(w, \mathbb{1}_{T_1(F),s}, \psi)$ has no poles in the region $\Re s > 0$, this shows $(\cdot,\cdot)_s$ is not zero for $s > 0$. This combined with the irreducibility of $I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$ shows that $(\cdot,\cdot)_s$ is non-degenerate for $0 \leq s < 1$. For $\phi \in I_{B_1}^{G_1}(\mathbb{1}_{T_1(F)})$, we have the associated "constant section"

$$\mathbb{C} \ni s \longmapsto \phi_s\Big(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k\Big) := \Big|\frac{a}{d}\Big|^{s/2} \phi(k) \in I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$$

of the "bundle" of induced representations $I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}) \to s \in \mathbb{C}$. The rationality of $N(\widetilde{w}, \mathbb{1}_{T_1(F),s})$ implies the continuity of $\nu_\phi : \mathbb{R}_{\geq 0} \ni s \mapsto (\phi_s, \phi_s)_s \in \mathbb{R}$ for any $\phi \in I_{B_1}^{G_1}(\mathbb{1}_{T_1(F)})$. Since

$I_{B_1}^{G_1}(\mathbb{1}_{T_1(F)})$ is unitarizable, Schur's lemma implies that $(\cdot,\cdot)_0$ must be, say, positive definite: $\nu_\phi(0) > 0$, $\forall\phi \neq 0, \in I_{B_1}^{G_1}(\mathbb{1}_{T_1(F)})$. The signature can be indefinite only after a zero of ν_ϕ or equivalently, of $N(\widetilde{w}, \mathbb{1}_{T_1(F),s})$, namely, a reducible point of $I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$. Hence $(\cdot,\cdot)_s$ is positive definite for $0 \leq s < 1$, and the representations in (3) of the theorem are unitarizable.

Finally, suppose that all the ν_ϕ, ($\phi \in I_{B_1}^{G_1}(\underline{\mathbb{1}})$) are still positive at some $s_0 > 1$. Then, since there is no reducible point in the region $s > s_0$, any $I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$, ($s > s_0$) is unitary with respect to the inner product $(\cdot,\cdot)_s$. We write $\mathcal{H}_{\mathbf{K}_1}(G_1(F))$ for the (unramified) *Hecke algebra* of $G_1(F)$, *i.e.*, the convolution algebra of compactly supported bi-$\mathbf{K}_1$-invariant functions on $G_1(F)$. Recall the *Satake isomorphism*

$$\mathcal{H}_{\mathbf{K}_1}(G_1(F)) \ni f \longmapsto \widehat{f}(q_F^{-s_1}, q_F^{-s_2}) := \mathrm{tr} I_{B_1}^{G_1}(|\ |_F^{s_1} \otimes |\ |_F^{s_2})(f) \in S[q_F^{-s_1}, q_F^{-s_2}],$$

where $S(X,Y)$ denotes the algebra of symmetric polynomials in X, Y. We take $f \in \mathcal{H}_{\mathbf{K}_1}(G_1(F))$ such that $\widehat{f}(X,Y) = X + Y$. Write $f = \sum_{i=1}^n a_i 1_{\mathbf{K}_1 g_i \mathbf{K}_1}$, where $1_{\mathbf{K}_1 g_i \mathbf{K}_1}$ is the characteristic function of the double coset $\mathbf{K}_1 g_i \mathbf{K}_1 \subset G_1(F)$. $\|\ \|_s$ denotes the norm with respect to the inner product $(\cdot,\cdot)_s$ on $I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$. The Iwasawa decomposition $G_1(F) = B_1(F)\mathbf{K}_1$ implies that the space $I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})^{\mathbf{K}_1}$ of $\mathbf{K}_1$-fixed vectors in $I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$ is 1-dimensional. For $\phi \in I_{B_1}^{G_1}(\mathbb{1}_{T_1(F)})^{\mathbf{K}_1}$, we have

$$\begin{aligned}|\mathrm{tr} I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}, f)| \|\phi_s\|_s = \|I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}, f)\phi_s\|_s = \Big\|\sum_{i=1}^n a_i I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}, 1_{\mathbf{K}_1 g_i \mathbf{K}_1})\phi_s\Big\|_s \\ \leq \sum_{i=1}^n |a_i| \|I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}, 1_{\mathbf{K}_1}) I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}, g_i) I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}, 1_{\mathbf{K}_1})\phi_s\|_s \\ \leq \sum_{i=1}^n |a_i| \|I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}, g_i)\phi_s\|_s \\ = \|\phi_s\|_s \sum_{i=1}^n |a_i|,\end{aligned}$$

because $\|\ \|_s$ is $G_1(F)$-invariant. This together with our choice of f yields

$$q_F^{-s/2} + q_F^{s/2} = \widehat{f}(q_F^{s/2}, q_F^{-s/2}) = \mathrm{tr} I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s}, f) \leq \sum_{i=1}^n |a_i|$$

for all $s > s_0$. This is absurd so that none of $I_{B_1}^{G_1}(\mathbb{1}_{T_1(F),s})$, ($s > 1$) is unitarizable. □

We go back to a general connected reductive group G over F for the moment. In the same spirit of this last argument, with the Hecke algebra $\mathcal{H}_{\mathbf{K}}(G(F))$ replaced by Bernstein's center [Ber84], one can prove the following theorem due to M. Tadić. (The beautiful argument is included in the third section of the cited paper.)

Theorem 2.7 ([Tad88] Th.2.5). *Fix $P = MU \in \mathcal{F}_0$ and $\sigma \in \Pi_0(M(F))$. Then the set of $e^\lambda \in A_{\widehat{M}}$ such that $I_P^G(\sigma_\lambda)$ admits an irreducible unitarizable subquotient is compact.*

3 Reducibility of induced representations of $GSp_4(F)$

3.1 Setting

We now turn to the group of our principal concern, namely $G = GSp_4$:

$$G(R) = \left\{ g \in \mathbb{M}_4(R) \,\middle|\, \nu(g) := g\mathrm{Ad}\Big(\begin{pmatrix} & \mathbf{1}_2 \\ -\mathbf{1}_2 & \end{pmatrix}\Big){}^tg \in R^\times \right\}.$$

Here, $\nu : G \to \mathbb{G}_m$ is the *similitude norm.*

Standard parabolic subgroups We take $M_0 = A_0$ to be the diagonal subgroup

$$T = \{m_0(a_1, a_2; \nu) := \mathrm{diag}(a_1, a_2, \nu a_1^{-1}, \nu a_2^{-1}) \,|\, a_i, \nu \in \mathbb{G}_m\}.$$

Also let $B = TU_0 \in \mathcal{P}(T)$ be the Borel subgroup given by

$$U_0 := \left\{ \left(\begin{array}{cc|cc} 1 & * & * & * \\ 0 & 1 & * & * \\ \hline & & 1 & 0 \\ & & * & 1 \end{array}\right) \in G \right\}.$$

Then the subset $\mathcal{F}(B) \subset \mathcal{F}_0$ (§1.4) consists of B, $P_i = M_iU_i$, ($i = 1, 2$), and G, where

$$M_1 = \left\{ m_1(t, g) := \left(\begin{array}{cc|cc} t & & & \\ & a & & b \\ \hline & & \nu/t & \\ & c & & d \end{array}\right) \,\middle|\, \begin{array}{l} t \in \mathbb{G}_m,\ \nu = \det g \\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_1 \end{array} \right\},$$

$$U_1 = \left\{ \left(\begin{array}{cc|cc} 1 & y' & s & y \\ 0 & 1 & y & 0 \\ \hline & & 1 & 0 \\ & & -y' & 1 \end{array}\right) \,\middle|\, y, y', s \in \mathbb{G}_a \right\},$$

$$M_2 = \left\{ m_2(g, \nu) := \left(\begin{array}{c|c} g & \\ \hline & \nu{}^tg^{-1} \end{array}\right) \,\middle|\, \begin{array}{c} g \in G_1 \\ \nu \in \mathbb{G}_m \end{array} \right\}, \quad U_2 = \left\{ \left(\begin{array}{c|c} \mathbf{1}_2 & S \\ \hline & \mathbf{1}_2 \end{array}\right) \,\middle|\, S = {}^tS \in \mathbb{M}_2 \right\}.$$

$\mathbb{M}_2$ stands for the 2×2 matrix algebra.

The group $X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$ of characters of T is given by $\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}\nu$, where $e_i(m_0(a_1, a_2; \nu)) = a_i$, ($i = 1, 2$). The set of B-positive roots of T is given by $\Sigma_B = \{\alpha_1 := e_1 - e_2, \alpha_2 := 2e_2 - \nu, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$. We write $r_i \in W$ for the reflection attached to α_i, ($i = 1, 2$). Then $W = \{1, r_1, r_2, r_1r_2, r_2r_1, w_{M_1} := r_1r_2r_1, w_{M_2} := r_2r_1r_2, w_0 := r_1r_2r_1r_2\}$ and $W(M_i) = \langle w_{M_i} \rangle$, ($i = 1, 2$). Also the group $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ of one-parameter subgroups of T admits a basis $\{e_1^\vee, e_2^\vee, \nu^\vee\}$ dual to $\{e_1, e_2, \nu\} \subset X^*(T)$. Here

$$e_1^\vee(t) = m_0(t, 1; 1), \quad e_2^\vee(t) = m_0(1, t; 1), \quad \nu^\vee(t) = m_0(1, 1; t).$$

Then the coroots of $\alpha \in \Sigma_B$ is given by $\alpha_1^\vee = e_1^\vee - e_2^\vee$, $\alpha_2^\vee = e_2^\vee$, $(\alpha_1 + \alpha_2)^\vee = \alpha_1^\vee + 2\alpha_2^\vee$, $(2\alpha_1 + \alpha_2)^\vee = \alpha_1^\vee + \alpha_2^\vee$.

L-group To describe a suitable system of normalization factors for intertwining operators (cf. Prop.1.13), it is convenient to adopt the L-group formulation.

The *L-group* of G is the direct product ${}^LG = \widehat{G} \times W_F$ with $\widehat{G} = G(\mathbb{C})$. $\widehat{B} = \widehat{T}\widehat{U}_0$ denotes the Borel subgroup $B(\mathbb{C}) = T(\mathbb{C})U_0(\mathbb{C}) \subset G(\mathbb{C})$. Then, by definition, we have the following identification of the root datum $(X^*(\widehat{T}), \Sigma_{\widehat{B}}, X_*(\widehat{T}), \Sigma_{\widehat{B}}^\vee)$ of $\widehat{G}$ with the dual root datum $(X_*(T), \Sigma_B^\vee, X^*(T), \Sigma_B)$ of G:

$$\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{\nu} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} e_1^\vee \\ e_2^\vee \\ \nu^\vee \end{pmatrix}, \quad \begin{pmatrix} e_1^\vee \\ e_2^\vee \\ \nu^\vee \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \\ \hat{\nu} \end{pmatrix}.$$

Here, $\{\hat{e}_1, \hat{e}_2, \hat{\nu}\}$ denotes the basis of $X^*(\widehat{T})$ defined in the same way as $\{e_1, e_2, \nu\} \subset X^*(T)$. Since $\alpha_1^\vee = 2\hat{e}_2 - \hat{\nu}$, $\alpha_2^\vee = \hat{e}_1 - \hat{e}_2$ in this identification, we have the following identification of the L-group of $P = MU \in \mathcal{F}(B)$:

$$\begin{gathered} {}^LB = {}^LT \ltimes \widehat{U}_0 = B(\mathbb{C}) \times W_F, \quad {}^LP_1 = {}^LM_1 \ltimes \widehat{U}_1 = P_2(\mathbb{C}) \times W_F, \\ {}^LP_2 = {}^LM_2 \ltimes \widehat{U}_2 = P_1(\mathbb{C}) \times W_F. \end{gathered}$$

The problem Let $P = MU \in \mathcal{F}(B)$. For $\pi \in \mathrm{Irr}(M(F))$, we can take $(\Re\pi)_G \in \mathfrak{a}_G^* = \mathbb{R}$ such that $|\omega_\pi(a\mathbf{1}_4)| = |a|_F^{(\Re\pi)_G}$, for any $a\mathbf{1}_4$ in the center $Z(F)$ of $G(F)$. Then the central character of $\pi^1 := |\nu|_F^{-(\Re\pi)_G/2} \otimes \pi$ restricted to $Z(F)$ is unitary, and $I_P^G(\pi) \simeq |\nu|_F^{(\Re\pi)_G/2} \otimes I_P^G(\pi^1)$. Thus granting Prop.1.8, in order to describe $\mathrm{Irr}(G(F)) \smallsetminus \mathrm{Irr}_0(G(F))$, it suffices to solve the following.

Problem 3.1. *Describe the composition series of the induced representations $I_P^G(\pi)$, where $P = MU$ is one of B, P_1, P_2 and $\pi \in \mathrm{Irr}_0(M(F))$ is of the following form:*

$$\begin{gathered} \boldsymbol{\chi}_{\boldsymbol{\lambda}} : T(F) \ni m_0(a_1, a_2; \nu) \longmapsto \chi_1(a_1)|a_1|_F^{\lambda_1}\chi_2(a_2)|a_2|_F^{\lambda_2}\chi(\nu)|\nu|_F^{-(\lambda_1+\lambda_2)/2} \in \mathbb{C}^\times, \\ \chi[\lambda] \otimes \tau[-\lambda/2] : M_1(F) \ni m_1(t, g) \longmapsto \chi(t)|t|_F^{\lambda}|\det g|_F^{-\lambda/2}\tau(g) \in GL(V_\tau), \\ \tau[\lambda] \otimes \chi[-\lambda] : M_2(F) \ni m_2(g, \nu) \longmapsto \chi(\nu)|\nu|_F^{-\lambda}|\det g|_F^{\lambda}\tau(g) \in GL(V_\tau). \end{gathered}$$

Here $\boldsymbol{\chi} = \chi_1 \otimes \chi_2 \otimes \chi \in \Pi(T(F))$, $\chi \in \Pi(F^\times)$, $\tau \in \Pi_0(G_1(F))$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ with $\lambda_1 \geq \lambda_2 \geq 0$, and $\lambda \in \mathbb{R}_{\geq 0}$.

3.2 Reducible points of generalized principal series

Here we determine the set of $\boldsymbol{\chi}_{\boldsymbol{\lambda}} \in \mathrm{Irr}_0(T(F))$ as in Prob.3.1 where $I_B^G(\boldsymbol{\chi}_{\boldsymbol{\lambda}})$ is reducible. We need the following normalization of intertwining operators $J_{B'|B}(\sigma_\lambda)$, $B' \in \mathcal{P}(T)$ [KS88]. For brevity, we write

$$\lambda_\pm := \frac{\lambda_1 \pm \lambda_2}{2}.$$

Let $W(\widehat{G}, \widehat{T})$ be the Weyl group of $\widehat{T}$ in $\widehat{G}$. Writing $\hat{r}_i \in W(\widehat{G}, \widehat{T})$ for the reflection associated to $\alpha_i^\vee \in \Sigma_{\widehat{B}} = \Sigma_B^\vee$, $(i = 1, 2)$, we have the identification of W with $W(\widehat{G}, \widehat{T})$ in which r_i corresponds to $\hat{r}_i$, $(i = 1, 2)$. Writing $B' = w(B)$ for some $w \in W$, we set

$\widehat{B}' = \widehat{T}\widehat{U}'_0 := w(\widehat{B})$. We write $\widehat{\mathfrak{u}}_0$, $\widehat{\mathfrak{u}}'_0$ for the Lie algebra of $\widehat{U}_0$, $\widehat{U}'_0$, respectively, and $\rho_{B'|B}$ for the adjoint representation of ${}^LT$ on $\widehat{\mathfrak{u}}'_0/\widehat{\mathfrak{u}}_0 \cap \widehat{\mathfrak{u}}'_0$. Identifying each $\chi \in \mathrm{Irr}(F^\times)$ with a quasi-character of W_F by the local classfield theory, the Langlands parameter of $\boldsymbol{\chi}_{\boldsymbol{\lambda}}$ as in Prob.3.1 is given by

$$\varphi_{\boldsymbol{\chi}_{\boldsymbol{\lambda}}} = \begin{pmatrix} \chi_1\chi_2\chi[\lambda_+] & & & \\ & \chi_1\chi[\lambda_-] & & \\ & & \chi[-\lambda_+] & \\ & & & \chi_2\chi[-\lambda_-] \end{pmatrix}.$$

The Artin L and ε-factors $L(s, \rho^\vee_{B'|B} \circ \varphi_{\boldsymbol{\chi}_{\boldsymbol{\lambda}}})$, $\varepsilon(s, \rho^\vee_{B'|B} \circ \varphi_{\boldsymbol{\chi}_{\boldsymbol{\lambda}}}, \psi)$ for the abelian representation $\rho^\vee_{B'|B} \circ \varphi_{\boldsymbol{\chi}_{\boldsymbol{\lambda}}}$ is defined as a product of the factors defined in [Tat79, (3.2)]. ($\rho^\vee_{B'|B}$ denotes the dual representation of $\rho_{B'|B}$.) We define

$$r_{B'|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}, \psi) := \frac{L(0, \rho^\vee_{B'|B} \circ \varphi_{\boldsymbol{\chi}_{\boldsymbol{\lambda}}})\varepsilon(0, \rho^\vee_{B'|B} \circ \varphi_{\boldsymbol{\chi}_{\boldsymbol{\lambda}}}, \psi)}{L(1, \rho^\vee_{B'|B} \circ \varphi_{\boldsymbol{\chi}_{\boldsymbol{\lambda}}})}.$$

Since $r_i(\mathfrak{u}_0)/\mathfrak{u}_0 \cap r_i(\mathfrak{u}_0)$ is identified with the root space of $-\alpha_i^\vee$ in the Lie algebra of $\widehat{G}$, we have

$$\begin{aligned} r_{r_i(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}, \psi) =& \frac{L(0, \boldsymbol{\chi}_{\boldsymbol{\lambda}} \circ \alpha_i^\vee)\varepsilon(0, \boldsymbol{\chi}_{\boldsymbol{\lambda}} \circ \alpha_i^\vee, \psi)}{L(1, \boldsymbol{\chi}_{\boldsymbol{\lambda}} \circ \alpha_i^\vee)} \\ =& \begin{cases} \dfrac{L(2\lambda_-, \chi_1\chi_2^{-1})\varepsilon(2\lambda_-, \chi_1\chi_2^{-1}, \psi)}{L(2\lambda_- + 1, \chi_1\chi_2^{-1})} & i = 1, \\ \dfrac{L(\lambda_2, \chi_2)\varepsilon(\lambda_2, \chi_2, \psi)}{L(\lambda_2 + 1, \chi_2)} & i = 2. \end{cases} \end{aligned} \tag{3.1}$$

For general $w \in W$, we take a reduced expression $w = r_{i_1} \cdots r_{i_\ell}$ by simple reflections and write $w_k := r_{i_1} \cdots r_{i_k}$, $(1 \le k \le \ell)$. Then we have a natural isomorphism obtained by taking the successive quotients:

$$w(\widehat{\mathfrak{u}}_0)/w(\widehat{\mathfrak{u}}_0) \cap \widehat{\mathfrak{u}}_0 \simeq \bigoplus_{k=1}^{\ell} w_k(\widehat{\mathfrak{u}}_0)/(w_k(\widehat{\mathfrak{u}}_0) \cap w_{k-1}(\widehat{\mathfrak{u}}_0))$$

noting $\mathrm{Ad}(w_{k-1})r_{i_k} \cdot w_{k-1} = w_{k-1}r_{i_k} = w_k$,

$$\begin{aligned} &= \bigoplus_{k=1}^{\ell} \mathrm{Ad}(w_{k-1})r_{i_k} \cdot w_{k-1}(\widehat{\mathfrak{u}}_0)/(\mathrm{Ad}(w_{k-1})r_{i_k} \cdot w_{k-1}(\widehat{\mathfrak{u}}_0) \cap w_{k-1}(\widehat{\mathfrak{u}}_0)) \\ &= \bigoplus_{k=1}^{\ell} w_{k-1}\Big(r_{i_k}(\widehat{\mathfrak{u}}_0)/r_{i_k}(\widehat{\mathfrak{u}}_0) \cap \widehat{\mathfrak{u}}_0\Big). \end{aligned}$$

Consequently,

$$r_{w(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}, \psi) = \prod_{k=1}^{\ell} r_{r_{i_k}(B)|B}(w_{k-1}^{-1}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}), \psi). \tag{3.2}$$

Lemma 3.2. *The normalized intertwining operators* $R_{B'|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) := r_{B'|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}, \psi)^{-1} J_{B'|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}})$ *satisfies the conditions of Prop.*1.13.

Proof. The property Prop.1.13 (iv) follows from the definition. The functional equation (ii) under the length condition follows from (1.5) and (3.2). Thanks to this partial result, we have to verify (i) only in the case $w = r_i$ $(i = 1, 2)$. But by virtue of (iv) and (1.4) applied to

$$\begin{aligned} I_B^G(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \cong & I_{P_2}^G(I_{B_1}^{G_1}(\chi_1[\lambda_1] \otimes \chi_2[\lambda_2]) \otimes \chi[-\lambda_+]) \\ \cong & I_{P_1}^G(\chi_1[\lambda_1] \otimes I_{B_1}^{G_1}(\chi_2\chi[-\lambda_-] \otimes \chi[-\lambda_+])), \end{aligned}$$

it suffices to prove the assertion for

$$r_{r_1(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}, \psi)^{-1} J_{\bar{B}_1|B_1}(\chi_1[\lambda_1] \otimes \chi_2[\lambda_2]), \quad r_{r_2(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}, \psi)^{-1} J_{\bar{B}_1|B_1}(\chi_2\chi[-\lambda_-] \otimes \chi[-\lambda_+]).$$

But this follows immediately from (2.7) (and (2.5)). The functional equation in the general case also reduces to those of operators on G_1 (2.9). The adjonction formula (iii) follows in the same way from the G_1-case, which can be deduced from (2.7). The statement (v) also follows from the same formula. We omit the details. □

Proposition 3.3. *For any* $\boldsymbol{\chi} = \chi_1 \otimes \chi_2 \otimes \chi \in \Pi(T(F))$, $I_B^G(\boldsymbol{\chi})$ *is irreducible.*

Proof. Thanks to Th.1.14, it suffices to show that

$$R_{B|B}(w, \boldsymbol{\chi}) = A(\boldsymbol{\chi}_w^+) \circ R_{w^{-1}(B)|B}(\boldsymbol{\chi}) = \boldsymbol{\chi}_w^+(\widetilde{w})\widetilde{w} \circ R_{w^{-1}(B)|B}(\boldsymbol{\chi})$$

is a scalar operator for any $w \in W_{\boldsymbol{\chi}} = \{w \in W \,|\, w(\boldsymbol{\chi}) = \boldsymbol{\chi}\}$. Taking a reduced expression $w = r_{i_\ell} \cdots r_{i_1}$, we proceed by the induction on ℓ. The case $\ell = 0$ is trivial. Setting $w_k := r_{i_k} \cdots r_{i_1}$, $(1 \le k \le \ell)$, we have $\widetilde{w} = t\widetilde{r}_{i_\ell}\widetilde{w}_{\ell-1}$ for some $t \in T(F)$. Thus

$$\begin{aligned} R_{B|B}(w, \boldsymbol{\chi}) = & \boldsymbol{\chi}_w^+(\widetilde{w}) t\widetilde{r}_{i_\ell}\widetilde{w}_{\ell-1} \circ R_{w(B)|w^{\ell-1}(B)}(\boldsymbol{\chi}) \circ R_{w^{\ell-1}(B)|B}(\boldsymbol{\chi}) \\ = & \boldsymbol{\chi}_w^+(\widetilde{w}t)\delta_{w(B)}(t)^{1/2}\widetilde{r}_{i_\ell} \circ R_{r_{i_\ell}(B)|B}(\boldsymbol{\chi}) \circ \Big(\widetilde{w}_{\ell-1} \circ R_{w_{\ell-1}(B)|B}(\boldsymbol{\chi})\Big). \end{aligned}$$

Inside the brace on the right hand side is a scalar operator by the induction hypothesis. On the other hand, by Prop.1.13 (iv),

$$\widetilde{r}_{i_\ell} \circ R_{r_{i_\ell}(B)|B}(\boldsymbol{\chi}) = \begin{cases} \Big(\widetilde{w} \circ J_{\bar{B}_1|B_1}(\chi_1 \otimes \chi_2)\Big) \otimes \mathrm{id}_\chi & \text{if } i_\ell = 1, \\ \mathrm{id}_{\chi_1} \otimes \Big(\widetilde{w} \circ J_{\bar{B}_1|B_1}(\chi_2\chi \otimes \chi)\Big) & \text{if } i_\ell = 2, \end{cases}$$

in the notation of §2. We consider only the case $i_\ell = 1$, since the other case is similar. The operator on $G_1(F)$ in the above is, modulo the measure factor $d(\mathrm{Ad}(\widetilde{w})\bar{u})/d\bar{u} \in \mathbb{R}_+^\times$, the normalized operator

$$N(\widetilde{w}, \chi_1 \otimes \chi_2)) = r(w, \chi_1 \otimes \chi_2, \psi)^{-1} M(\widetilde{w}, \chi_1 \otimes \chi_2) : I_{B_1}^{G_1}(\chi_1 \otimes \chi_2) \longrightarrow I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)$$

introduced in the proof of Th.2.6. Since $I_{B_1}^{G_1}(\chi_1, \chi_2)$ is irreducible (Th.2.3 (i)), this is a non-zero scalar operator. □

Next let us consider the case $\boldsymbol{\lambda} \neq 0$.

Proposition 3.4. *$I_B^G(\boldsymbol{\chi_\lambda})$ in Prob.*3.1 *is reducible if and only if $\boldsymbol{\chi_\lambda}$ lies in the union of the following lines:*

$$\mathfrak{r}_{\alpha_1}(\chi_1) := \{\boldsymbol{\chi_\lambda} \,|\, \chi_1 = \chi_2,\, \lambda_1 - \lambda_2 = 1\}, \quad \mathfrak{r}_{\alpha_2}(\chi_1) := \{\boldsymbol{\chi_\lambda} \,|\, \chi_2 = \mathbb{1},\, \lambda_2 = 1\},$$
$$r_2(\mathfrak{r}_{\alpha_1}(\chi_1)) := \{\boldsymbol{\chi_\lambda} \,|\, \chi_1 = \chi_2^{-1},\, \lambda_1 + \lambda_2 = 1\}, \quad r_1(\mathfrak{r}_{\alpha_2}(\chi_2)) := \{\boldsymbol{\chi_\lambda} \,|\, \chi_1 = \mathbb{1},\, \lambda_1 = 1\}.$$

Proof. We first prove that $I_B^G(\boldsymbol{\chi_\lambda})$ is reducible if and only if $R_{\bar{B}|B}(\boldsymbol{\chi_\mu})$ has a zero at $\boldsymbol{\mu} = \boldsymbol{\lambda}$. In fact, if $\lambda_1 > \lambda_2 > 0$, $I_B^G(\boldsymbol{\chi_\lambda})$ is reducible if and only if $J_{\bar{B}|B}(\boldsymbol{\chi_\mu})$ has a zero at $\boldsymbol{\mu} = \boldsymbol{\lambda}$ by Th.1.11. Since $r_{\bar{B}|B}(\boldsymbol{\chi_\lambda}, \psi)$ has neither poles nor zeros in this region, we may replace $J_{\bar{B}|B}(\boldsymbol{\chi_\mu})$ with $R_{\bar{B}|B}(\boldsymbol{\chi_\mu})$. Next if $\lambda_1 = \lambda_2 > 0$, we have $I_B^G(\boldsymbol{\chi_\lambda}) \simeq I_{P_2}^G(I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)[\lambda_1] \otimes \chi[-\lambda_1])$. Thus by the Langlands classification as in the first case, this is reducible if and only if $J_{\bar{P}_2|P_2}(I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)[\lambda_1] \otimes \chi[-\lambda_1])$, or equivalently $R_{\bar{P}_2|P_2}(I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)[\lambda_1] \otimes \chi[-\lambda_1])$ admits a non-trivial kernel. But since $\bar{U}_2(\bar{U}_0 \cap M_2) = \bar{U}_0$, we have

$$\begin{aligned} J_{\bar{B}|B}(\boldsymbol{\chi_\mu}) = J_{\bar{P}_2|P_2}(I_{B_1}^{G_1}(\chi_1[\mu_1] \otimes \chi_2[\mu_2]) \otimes \chi[-\mu_+]) \\ \circ J_{\bar{B}_1|B_1}(\chi_1[\mu_1] \otimes \chi_2[\mu_2]) \otimes \mathrm{id}_{\chi[-\mu_+]}, \end{aligned}$$

for $\mu_1 > \mu_2 > 0$. Then our choice of normalization factor yields

$$\begin{aligned} R_{\bar{B}|B}(\boldsymbol{\chi_\mu}) = R_{\bar{P}_2|P_2}(I_{B_1}^{G_1}(\chi_1[\mu_1] \otimes \chi_2[\mu_2]) \otimes \chi[-\mu_+]) \\ \circ R_{\bar{B}_1|B_1}(\chi_1[\mu_1] \otimes \chi_2[\mu_2]) \otimes \mathrm{id}_{\chi[-\mu_+]}, \end{aligned}$$

for any $\boldsymbol{\mu} \in \mathbb{R}^2$. Thus, $R_{\bar{P}_2|P_2}(I_{B_1}^{G_1}(\chi_1 \otimes \chi_2)[\lambda_1] \otimes \chi[-\lambda_1])$ admits a non-trivial kernel if and only if so does $R_{\bar{B}|B}(\boldsymbol{\chi_\lambda})$, since $R_{\bar{B}_1|B_1}(\chi_1 \otimes \chi_2)$ is a non-zero scalar operator (Th.2.3). Finally if $\lambda_2 = 0$, $I_B^G(\boldsymbol{\chi_\lambda})$ can be written as $I_{P_1}^G(\chi_1[\lambda_1] \otimes I_{B_1}^{G_1}(\chi_2\chi, \chi)[-\lambda_1/2])$. We apply the similar argument as above to

$$\begin{aligned} R_{\bar{B}|B}(\boldsymbol{\chi_\mu}) = R_{\bar{P}_1|P_1}(\chi_1[\mu_1] \otimes I_{B_1}^{G_1}(\chi_2\chi[-\mu_-] \otimes \chi[-\mu_+]) \\ \circ \mathrm{id}_{\chi_1[\mu_1]} \otimes R_{\bar{B}_1|B_1}(\chi_2\chi[-\mu_-] \otimes \chi[-\mu_+]). \end{aligned}$$

The assertion follows because $R_{\bar{B}_1|B_1}(\chi_2\chi \otimes \chi)$ is a non-zero scalar.

Now we appeal to the functional equation. Noting $\bar{B} = w_0^{-1}(B)$, we take the reduced expression $w_0 = r_2r_1r_2r_1$. Then we have

$$\begin{aligned} R_{\bar{B}|B}(\boldsymbol{\chi_\lambda}) =& R_{w_0^{-1}(B)|w_{M_1}^{-1}(B)}(\boldsymbol{\chi_\lambda}) \circ R_{w_{M_1}^{-1}(B)|r_1r_2(B)}(\boldsymbol{\chi_\lambda}) \\ & \circ R_{r_1r_2(B)|r_1(B)}(\boldsymbol{\chi_\lambda}) \circ R_{r_1(B)|B}(\boldsymbol{\chi_\lambda}) \\ =& \widetilde{w}_{M_1}^{-1} \circ R_{r_2(B)|B}(w_{M_1}(\boldsymbol{\chi_\lambda})) \circ \widetilde{w}_{M_1} \circ (\widetilde{r_2r_1})^{-1} \circ R_{r_1(B)|B}(r_2r_1(\boldsymbol{\chi_\lambda})) \circ (\widetilde{r_2r_1}) \\ & \circ \widetilde{r}_1^{-1} \circ R_{r_2(B)|B}(r_1(\boldsymbol{\chi_\lambda})) \circ \widetilde{r}_1 \circ R_{r_1(B)|B}(\boldsymbol{\chi_\lambda}), \end{aligned}$$

so that this has a non-trivial kernel if and only if so does any of the following operators:

$$R_{r_1(B)|B}(\boldsymbol{\chi_\lambda}) = R_{\bar{B}_1|B_1}(\chi_1[\lambda_1] \otimes \chi_2[\lambda_2]) \otimes \mathrm{id}_{\chi[-\lambda_+]},$$
$$R_{r_2(B)|B}(r_1(\boldsymbol{\chi_\lambda})) = \mathrm{id}_{\chi_2[\lambda_2]} \otimes R_{\bar{B}_1|B_1}(\chi_1\chi[\lambda_-] \otimes \chi[-\lambda_+]),$$
$$R_{r_1(B)|B}(r_2r_1(\boldsymbol{\chi_\lambda})) = R_{\bar{B}_1|B_1}(\chi_2[\lambda_2] \otimes \chi_1^{-1}[-\lambda_1]) \otimes \mathrm{id}_{\chi_1\chi[\lambda_-]},$$
$$R_{r_2(B)|B}(w_{M_1}(\boldsymbol{\chi_\lambda})) = \mathrm{id}_{\chi_1^{-1}[-\lambda_1]} \otimes R_{\bar{B}_1|B_1}(\chi_1\chi_2\chi[\lambda_+] \otimes \chi_1\chi[\lambda_-]).$$

Combining Th.1.11 with Th.2.3, we see that the set of zeros of these operators are $\mathfrak{r}_{\alpha_1}(\chi_1)$, $r_1(\mathfrak{r}_{\alpha_2}(\chi_2))$, $r_2(\mathfrak{r}_{\alpha_1}(\chi_1))$, $\mathfrak{r}_{\alpha_2}(\chi_1)$, respectively. $\square$

4 Classification of $\mathrm{Irr}(G(F)) \smallsetminus \mathrm{Irr}_0(G(F))$

4.1 Further reducibility of $I_B^G(\boldsymbol{\chi_\lambda})$

It follows from Th.2.3 and the induction by stages (§1.2 (iii)) that we have the following exact sequences of $G(F)$-modules on the reducible lines in Prop.3.4.

(i) On $\mathfrak{r}_{\alpha_1}(\chi_1)$,

$$I_{P_2}^G(\chi_1[\lambda_+](\det)\delta^{G_1} \otimes \chi[-\lambda_+]) \hookrightarrow I_B^G(\boldsymbol{\chi_\lambda}) \twoheadrightarrow I_{P_2}^G(\chi_1[\lambda_+](\det) \otimes \chi[-\lambda_+]).$$

(ii) On $\mathfrak{r}_{\alpha_2}(\chi_1)$,

$$I_{P_1}^G(\chi_1[\lambda_1] \otimes \chi[-\lambda_1/2](\det)\delta^{G_1}) \hookrightarrow I_B^G(\boldsymbol{\chi_\lambda}) \twoheadrightarrow I_{P_1}^G(\chi_1[\lambda_1] \otimes \chi[-\lambda_1/2](\det)).$$

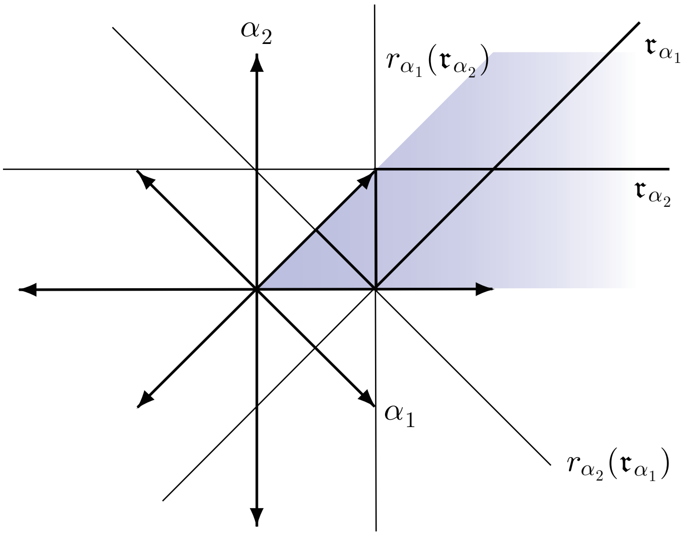
(iii) On $r_2(\mathfrak{r}_{\alpha_1}(\chi_1))$ with $1/2 \le \lambda_1 < 1$ if $\chi_1 = \mathbb{1}_{F^\times}$, and $1/2 \le \lambda_1 \le 1$ otherwise,

$$I_{P_2}^G(\chi_1[\lambda_-](\det)\delta^{G_1} \otimes \chi_1^{-1}\chi[-\lambda_-]) \hookrightarrow I_B^G(\boldsymbol{\chi_\lambda}) \twoheadrightarrow I_{P_2}^G(\chi_1[\lambda_-](\det) \otimes \chi_1^{-1}\chi[-\lambda_-]).$$

(iv) On $r_1(\mathfrak{r}_{\alpha_2}(\chi_2))$ with $0 < \lambda_2 \le 1$ if $\chi_2 = \mathbb{1}_{F^\times}$, and $0 \le \lambda_2 \le 1$ otherwise,

$$I_{P_1}^G(\chi_2[\lambda_2] \otimes \chi[-\lambda_2/2](\det)\delta^{G_1}) \hookrightarrow I_B^G(\boldsymbol{\chi_\lambda}) \twoheadrightarrow I_{P_1}^G(\chi_2[\lambda_2] \otimes \chi[-\lambda_2/2](\det)).$$

Note that both of the removed points in (iii), (iv) in the cases $\chi_1 = \mathbb{1}_{F^\times}$, $\chi_2 = \mathbb{1}_{F^\times}$, respectively, are contained in $\mathfrak{r}_{\alpha_1}$ (see the picture on the right). In this subsection, we determine the possible reducible points of the constituents appeared in the left and right of the above sequences. As is evident from the picture, we have only to consider $I_{P_1}^G((\chi_1 \otimes \pi)_\lambda)$ and $I_{P_2}^G((\pi \otimes \chi)_\lambda)$ for $\lambda \ge 0$, where π is either one-dimensional or of Steinberg type.



Lemma 4.1 ([ST93] Lem.3.3, 3.4)**.** *(a) The constituents* $I_{P_1}^G(\chi_1[\lambda] \otimes \chi[-\lambda/2](\det)\delta^{G_1})$ *and* $I_{P_1}^G(\chi_1[\lambda] \otimes \chi[-\lambda/2](\det))$ *of* $I_B^G(\chi_1[\lambda] \otimes |\ |_F \otimes \chi[-(\lambda+1)/2])$ *with* $\lambda \ge 0$ *are irreducible except at the points*

$$\mathfrak{r}_{\alpha_2,0} := \mathbb{1}_{F^\times} \otimes |\ |_F \otimes \chi[-1/2], \quad \mathfrak{r}_{\alpha_2,1} := |\ |_F \otimes |\ |_F \otimes \chi[-1], \quad \mathfrak{r}_{\rho_B} := |\ |_F^2 \otimes |\ |_F \otimes |\ |_F^{-3/2}.$$

(b) The constituents $I^G_{P_2}(\chi_1[\lambda](\det)\delta^{G_1}\otimes\chi[-\lambda])$ *and* $I^G_{P_2}(\chi_1[\lambda](\det)\otimes\chi[-\lambda])$ *of* $I^G_B(\chi_1[\lambda+1/2]\otimes\chi_1[\lambda-1/2]\otimes\chi[-\lambda])$ *with* $\lambda\geq 0$ *are irreducible except at the points*

$$\mathfrak{r}_{\alpha_1,0}(E):=\omega_{E/F}[1/2]\otimes\omega_{E/F}[-1/2]\otimes\chi,\quad \mathfrak{r}_{\alpha_1,1/2}(E):=\omega_{E/F}[1]\otimes\omega_{E/F}\otimes\chi[-1/2],$$

and $\mathfrak{r}_{\rho_B}$ *as above. Here,* E *runs over the set of quadratic extensions of* F *in* $\bar{F}$*. Here we also include the case* $E=F^2$ *for brevity of the exposition.*

Proof. We only prove (b). The assertion (a) can be proved in the same way. We first consider the case $\lambda=0$.

In general, for any parabolic subgroup $P=MU$ and $\pi\in\Pi(M(F))$, the induced representation $I^G_P(\pi)$ is reducible if and only if $\dim\mathrm{End}_{G(F)}(I^G_P(\pi))>1$. It follows from the Frobenius reciprocity Prop.1.3 and Th.1.4 that

$$\begin{aligned}\dim\mathrm{End}_{G(F)}(I^G_P(\pi))&=\dim\mathrm{Hom}_{M(F)}(I^G_P(\pi)_P,\pi)\\&\leq\sum_{w\in{}_PW_P}\dim\mathrm{Hom}_{M(F)}(I^M_{w(P)\cap M}(w(\pi_{w^{-1}(P)\cap M})),\pi)\\&=\sum_{w\in{}_PW_P}\dim\mathrm{Hom}_{M(F)}(\pi^\vee,I^M_{w(P)\cap M}(w(\pi_{w^{-1}(P)\cap M})^\vee))\\&=\sum_{w\in{}_PW_P}\dim\mathrm{Hom}_{(w(M)\cap M)(F)}((\pi^\vee)_{w(P)\cap M},w(\pi_{w^{-1}(P)\cap M})^\vee)\\&=\sum_{w\in{}_PW_P}\dim\mathrm{Hom}_{(w(M)\cap M)(F)}(w(\pi_{w^{-1}(P)\cap M}),\pi_{w(\bar{P})\cap M}).\end{aligned}\tag{4.1}$$

We apply this to $P=P_2$ and $\pi\simeq\chi_1(\det)\pi_1\otimes\chi$, where $\pi_1\simeq\delta^{G_1}$ or $\mathbb{1}_{G_1(F)}$. One has ${}_{P_2}W_{P_2}=\{1,r_2,w_{M_2}\}$ and

$$\begin{cases}w^{-1}(P_2)\cap M_2=w(\bar{P}_2)\cap M_2=M_2 & \text{if } w=1,w_{M_2},\\ w^{-1}(P_2)\cap M_2=B\cap M_2,\ w(\bar{P}_2)\cap M_2=\bar{B}\cap M_2 & \text{if } w=r_2.\end{cases}$$

Thus the spaces associated to $1,w_{M_2},r_2$ become $\mathrm{End}_{M_2(F)}(\pi)$,

$$\mathrm{Hom}_{M_2(F)}(\chi_1^{-1}(\det)\pi_1\otimes\chi\chi_1^2,\chi_1(\det)\pi_1\otimes\chi),\quad\mathrm{Hom}_{T(F)}(\boldsymbol{\chi}_{\pm(1/2,1/2)},\boldsymbol{\chi}_{\mp(1/2,-1/2)}),$$

respectively. Thus $\dim\mathrm{End}_{G(F)}(I^G_{P_2}(\chi_1(\det)\pi_1\otimes\chi))$ can be greater than one only if the second space is not zero, namely $\chi_1^2=\mathbb{1}_{F^\times}$, $\boldsymbol{\chi}_{\boldsymbol{\lambda}}=\mathfrak{r}_{\alpha_1,0}$.

Next consider the case $I^G_{P_2}(\chi_1[\lambda](\det)\delta^{G_1}\otimes\chi[-\lambda])$ with $\lambda>0$. We write $\boldsymbol{\lambda}=(\lambda+1/2,\lambda-1/2)$, By the Langlands classification, the representation in question is reducible if and only if the kernel of the second row in the following diagram is not zero:

$$\begin{tikzcd}
I^G_B(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \arrow[r, "J_{w_{M_2}(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}})"] & I^G_{w_{M_2}(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \\
I^G_{P_2}(\chi_1[\lambda](\det)\delta^{G_1}\otimes\chi[-\lambda]) \arrow[u] \arrow[r, "J_{\bar{P}_2|P_2}(\chi_1[\lambda](\det)\delta^{G_1}\otimes\chi[-\lambda])"] & I^G_{\bar{P}_2}(\chi_1[\lambda](\det)\delta^{G_1}\otimes\chi[-\lambda]) \arrow[u]
\end{tikzcd}$$

Here, the vertical arrows are the injections in p.27 (i). For this, it is necessary that the operator in the first row:

$$\begin{aligned} J_{w_{M_2}(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) =& J_{w_{M_2}(B)|r_2r_1(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \circ J_{r_2r_1(B)|r_2(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \circ J_{r_2(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \\ =& r_2r_1\big(J_{r_2(B)|B}(r_1r_2(\boldsymbol{\chi}_{\boldsymbol{\lambda}}))\big) \circ r_2\big(J_{r_1(B)|B}(r_2(\boldsymbol{\chi}_{\boldsymbol{\lambda}}))\big) \circ J_{r_2(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \end{aligned}$$

has non-trivial kernel. On the other hand, (1.4) allows us to write the operators in the right hand side as

$$\begin{aligned} J_{r_2(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) &= \mathrm{id}_{\chi_1[\lambda+1/2]} \otimes J_{\bar{B}_1|B_1}(\chi_1\chi|\ |_F^{-1/2} \otimes \chi|\ |_F^{-\lambda}), \\ J_{r_1(B)|B}(r_2(\boldsymbol{\chi}_{\boldsymbol{\lambda}})) &= J_{\bar{B}_1|B_1}(\chi_1|\ |_F^{\lambda+1/2} \otimes \chi_1^{-1}|\ |_F^{1/2-\lambda}) \otimes \mathrm{id}_{\chi_1\chi[-1/2]}, \\ J_{r_2(B)|B}(r_1r_2(\boldsymbol{\chi}_{\boldsymbol{\lambda}})) &= \mathrm{id}_{\chi_1^{-1}[1/2-\lambda]} \otimes J_{\bar{B}_1|B_1}(\chi_1^2\chi|\ |_F^{\lambda} \otimes \chi_1\chi|\ |_F^{-1/2}). \end{aligned}$$

Each of these has non-trivial kernel if and only if $(\chi_1 = \mathbb{1}_{F^\times}, \lambda = 3/2)$, $(\chi_1^2 = \mathbb{1}_{F^\times}, \lambda = 1/2)$ and $(\chi_1 = \mathbb{1}_{F^\times}, \lambda = 1/2)$, respectively, as the lemma asserts.

Finally, we consider $I_{P_2}^G(\chi_1[\lambda](\det) \otimes \chi[-\lambda]) = \mathrm{im} R_{r_1(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}})$. First assume $\lambda \geq 1/2$. If $\lambda > 1/2$, the Langlands classification applied to $I_B^G(\boldsymbol{\chi}_{\boldsymbol{\lambda}})$ implies that the image of the normalized intertwining operator $R_{\bar{B}|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}})$ is irreducible. (Note that $r_{\bar{B}|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}, \psi)$ is holomorphic and non-zero at such $\boldsymbol{\lambda}$.) When $\lambda = 1/2$, $\boldsymbol{\lambda} = (1, 0)$, again Th.1.11 implies that the image of $R_{\bar{P}_1|P_1}(\chi_1[1] \otimes I_{B_1}^{G_1}(\chi_1\chi \otimes \chi)[-1/2])$ is irreducible. Identifying $I_{P_1}^G\big(\chi_1[1] \otimes I_{B_1}^{G_1}(\chi_1\chi \otimes \chi)[-1/2]\big) \simeq I_{r_2(B)}^G(\boldsymbol{\chi}_{\boldsymbol{\lambda}})$, $I_{\bar{P}_1}^G(\chi_1[1] \otimes I_{B_1}^{G_1}(\chi_1\chi \otimes \chi)[-1/2]) \simeq I_{\bar{B}}^G(\boldsymbol{\chi}_{\boldsymbol{\lambda}})$, respectively, we have the functional equation

$$R_{\bar{P}_1|P_1}\big(\chi_1[1] \otimes I_{B_1}^{G_1}(\chi_1\chi \otimes \chi)[-\frac{1}{2}]\big) \circ \big(\mathrm{id}_{\chi_1[1]} \otimes R_{\bar{B}_1|B_1}(\chi_1\chi \otimes \chi)[\frac{1}{2}]\big) = R_{\bar{B}|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}).$$

Since the latter operator in the left hand side is a scalar, we conclude that $\mathrm{im} R_{\bar{B}|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}})$ is irreducible for any $\lambda \geq 1/2$.

Now the functional equation

$$\begin{aligned} R_{\bar{B}|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) =& R_{w_0(B)|r_1(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \circ R_{r_1(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \\ =& r_1\Big(R_{w_{M_2}(B)|B}(r_1(\boldsymbol{\chi}_{\boldsymbol{\lambda}}))\Big) \circ R_{r_1(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}). \end{aligned}$$

implies that $I_{P_2}^G(\chi_1[\lambda](\det) \otimes \chi[-\lambda])$ is reducible if and only if the kernel of the operator

$$R_{w_{M_2}(B)|B}(r_1(\boldsymbol{\chi}_{\boldsymbol{\lambda}})) = r_2r_1\Big(R_{r_2(B)|B}(w_{M_1}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}))\Big) \circ r_2\Big(R_{r_1(B)|B}(r_2r_1(\boldsymbol{\chi}_{\boldsymbol{\lambda}}))\Big) \circ R_{r_2(B)|B}(r_1(\boldsymbol{\chi}_{\boldsymbol{\lambda}}))$$

is not zero. By Prop.1.13 (iv), the operators on the right hand side can be written as

$$\begin{aligned} R_{r_2(B)|B}(r_1(\boldsymbol{\chi}_{\boldsymbol{\lambda}})) &= \mathrm{id}_{\chi_1[\lambda-1/2]} \otimes R_{\bar{B}_1|B_1}(\chi_1\chi[\frac{1}{2}] \otimes \chi[-\lambda]), \\ R_{r_1(B)|B}(r_2r_1(\boldsymbol{\chi}_{\boldsymbol{\lambda}})) &= R_{\bar{B}_1|B_1}(\chi_1[\lambda - \frac{1}{2}] \otimes \chi_1[\lambda + \frac{1}{2}]^{-1}) \otimes \mathrm{id}_{\chi_1\chi[1/2]}, \\ R_{r_2(B)|B}(w_{M_1}(\boldsymbol{\chi}_{\boldsymbol{\lambda}})) &= \mathrm{id}_{\chi_1[\lambda+1/2]^{-1}} \otimes R_{\bar{B}_1|B_1}(\chi_1^2\chi[\lambda] \otimes \chi_1\chi[\frac{1}{2}]). \end{aligned}$$

Each of these has its unique zero in the region under consideration at $(\chi_1 = \mathbb{1}_{F^\times}, \lambda = 1/2)$, $(\chi_2^2 = \mathbb{1}_{F^\times}, \lambda = 1/2)$, $(\chi_1 = \mathbb{1}_{F^\times}, \lambda = 3/2)$, respectively.

Next consider the case $0 < \lambda < 1/2$. We have

$$I_{P_2}^G(\chi_1[\lambda](\det) \otimes \chi[-\lambda]) = \mathrm{im} J_{r_1(B)|B}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \circ J_{B|r_2(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) = \mathrm{im} J_{r_1(B)|r_2(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}),$$

because $J_{B|r_2(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) = \mathrm{id}_{\chi_1[\lambda+1/2]} \otimes J_{B_1|\bar{B}_1}(\chi_1\chi[-1/2] \otimes \chi[-\lambda])$ is an isomorphism for present λ. Since $I_{r_2(B)}^G(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) = r_2\big(I_B^G(r_2(\boldsymbol{\chi})_{(\lambda+1/2, 1/2-\lambda)})\big)$ is a standard module, the image of

$$J_{r_2(\bar{B})|r_2(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) = J_{w_{M_1}(B)|r_1r_2(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \circ J_{r_1r_2(B)|r_1(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) \circ J_{r_1(B)|r_2(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}})$$

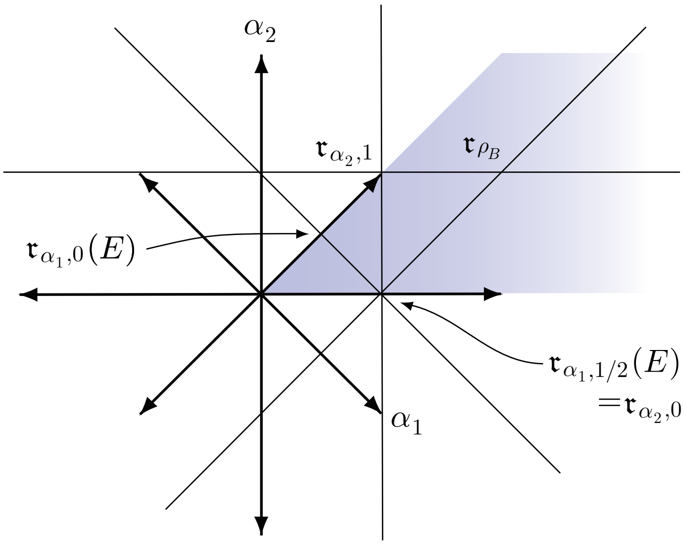
is irreducible by Th.1.11. Thus the reducibility in question occurs if and only if any of the operators

$$\begin{aligned} J_{r_1r_2(B)|r_1(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) &= r_1\Big(\mathrm{id}_{\chi_1[\lambda-1/2]} \otimes J_{\bar{B}_1|B_1}(\chi_1\chi[\frac{1}{2}] \otimes \chi[-\lambda])\Big), \\ J_{w_{M_1}(B)|r_1r_2(B)}(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) &= r_1r_2\Big(J_{\bar{B}_1|B_1}(\chi_1[\lambda - \frac{1}{2}] \otimes \chi_1[\lambda + \frac{1}{2}]^{-1}) \otimes \mathrm{id}_{\chi_1\chi[1/2]}\Big) \end{aligned}$$

admits non-trivial kernel. This does not occur in the present case. □

4.2 Irreducible representations supported on T

Now we get into the contribution of Sally-Tadić to the subject, namely, a description of the irreducible constituents of the representations in Lem.4.1 at the possible reducible points $\mathfrak{r}_{\alpha_2,0}$, $\mathfrak{r}_{\alpha_2,1}$, $\mathfrak{r}_{\alpha_1,0}(E)$, $\mathfrak{r}_{\alpha_1,1/2}(E)$ and $\mathfrak{r}_{\rho_B}$. For this, we need to study the cuspidal Jacquet modules (*i.e.*, Jacquet modules along B) of the relevant representations. The real part of the above points are illustrated in the picture on the right.
We first state the result.



Theorem 4.2. *For* $\boldsymbol{\chi} = \chi_1 \otimes \chi_2 \otimes \chi \in \Pi(T(F))$ *and* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ *with* $\lambda_1 \geq \lambda_2 \geq 0$, *the composition series of* $I_B^G(\boldsymbol{\chi}_{\boldsymbol{\lambda}}) = I_B^G(\chi_1|\ |_F^{\lambda_1} \otimes \chi_2|\ |_F^{\lambda_2} \otimes \chi|\ |_F^{-(\lambda_1+\lambda_2)/2})$ *is given as follows.*
(1) At $\mathfrak{r}_{\rho_B} = |\ |_F^2 \otimes |\ |_F \otimes \chi|\ |_F^{-3/2}$, *we have*

$$\begin{aligned} &0 \longrightarrow I_{P_1}^G(|\ |_F^2 \otimes \chi(\det)\delta^{G_1}[-1]) \longrightarrow I_B^G(\mathfrak{r}_{\rho_B}) \longrightarrow I_{P_1}^G(|\ |_F^2 \otimes \chi(\det)[-1]) \longrightarrow 0 \\ &0 \longrightarrow I_{P_2}^G(\delta^{G_1}[3/2] \otimes \chi|\ |_F^{-3/2}) \longrightarrow I_B^G(\mathfrak{r}_{\rho_B}) \longrightarrow I_{P_2}^G(|\det|_F^{3/2} \otimes \chi|\ |_F^{-3/2}) \longrightarrow 0, \end{aligned} \tag{4.2.0}$$

and

$$\begin{gathered}
0 \longrightarrow \chi(\nu)\delta_0^G \longrightarrow I_{P_1}^G(|\ |_F^2 \otimes \chi(\det)\delta^{G_1}[-1]) \longrightarrow J_{P_1}^G(|\ |_F^2 \otimes \chi(\det)\delta^{G_1}[-1]) \longrightarrow 0, \\
0 \longrightarrow \chi(\nu)\delta_0^G \longrightarrow I_{P_2}^G(\delta^{G_1}[3/2] \otimes \chi|\ |_F^{-3/2}) \longrightarrow J_{P_2}^G(\delta^{G_1}[3/2] \otimes \chi|\ |_F^{-3/2}) \longrightarrow 0, \\
0 \longrightarrow J_{P_2}^G(\delta^{G_1}[3/2] \otimes \chi|\ |_F^{-3/2}) \longrightarrow I_{P_1}^G(|\ |_F^2 \otimes \chi(\det)[-1]) \longrightarrow \chi(\nu) \longrightarrow 0, \\
0 \longrightarrow J_{P_1}^G(|\ |_F^2 \otimes \chi(\det)\delta^{G_1}[-1]) \longrightarrow I_{P_2}^G(|\det|_F^{3/2} \otimes \chi|\ |_F^{-3/2}) \longrightarrow \chi(\nu) \longrightarrow 0.
\end{gathered} \tag{4.2.1}$$

Here $\delta_0^G \in \Pi_{\mathrm{disc}}(G(F))$ *is the* Steinberg representation *of* $G(F)$.

(2) At $\mathfrak{r}_{\alpha_1,1/2}(E) = \omega_{E/F}|\ |_F \otimes \omega_{E/F} \otimes \chi|\ |_F^{-1/2}$ *for a quadratic extension* E *of* F*, we have*

$$I_{P_2}^G(\omega_{E/F}(\det)\delta^{G_1}[\frac{1}{2}] \otimes \chi|\ |_F^{-1/2}) \hookrightarrow I_B^G(\mathfrak{r}_{\alpha_1,1/2}(E)) \twoheadrightarrow I_{P_2}^G(\omega_{E/F}(\det)[\frac{1}{2}] \otimes \chi|\ |_F^{-1/2}),$$

and

$$\begin{aligned}
&\chi(\nu)\delta_0^G(E) \hookrightarrow I_{P_2}^G(\omega_{E/F}(\det)\delta^{G_1}[\frac{1}{2}] \otimes \chi|\ |_F^{-1/2}) \twoheadrightarrow J_{P_2}^G(\omega_{E/F}(\det)\delta^{G_1}[\frac{1}{2}] \otimes \chi|\ |_F^{-1/2}), \\
&J_{P_2}^G(\omega_{E/F}(\det)\delta^{G_1}[\frac{1}{2}] \otimes \chi\omega_{E/F}|\ |_F^{-1/2}) \hookrightarrow I_{P_2}^G(\omega_{E/F}(\det)[\frac{1}{2}] \otimes \chi|\ |_F^{-1/2}) \\
&\qquad\qquad\qquad\qquad \twoheadrightarrow J_{P_1}^G(\omega_{E/F}|\ |_F \otimes I_{B_1}^{G_1}(\omega_{E/F}\chi \otimes \chi)[-\frac{1}{2}]).
\end{aligned} \tag{4.2.2}$$

Here $\delta_0^G(E) \in \Pi_{\mathrm{disc}}(G(F))$.

(3) At $\mathfrak{r}_{\alpha_1,1/2}(F^2) = |\ |_F \otimes \mathbb{1}_{F^\times} \otimes \chi|\ |_F^{-1/2}$*, we have*

$$\begin{gathered}
0 \longrightarrow I_{P_1}^G(\mathbb{1}_{F^\times} \otimes \chi(\det)\delta^{G_1}) \longrightarrow I_B^G(\mathfrak{r}_{\alpha_1,1/2}(F^2)) \longrightarrow I_{P_1}^G(\mathbb{1}_{F^\times} \otimes \chi(\det)) \longrightarrow 0, \\
0 \longrightarrow I_{P_2}^G(\delta^{G_1}[1/2] \otimes \chi|\ |_F^{-1/2}) \longrightarrow I_B^G(\mathfrak{r}_{\alpha_1,1/2}(F^2)) \longrightarrow I_{P_2}^G(|\det|_F^{1/2} \otimes \chi|\ |_F^{-1/2}) \longrightarrow 0,
\end{gathered}$$

and

$$\begin{gathered}
I_{P_1}^G(\mathbb{1}_{F^\times} \otimes \chi(\det)\delta^{G_1}) = \chi(\nu)\tau_0(\delta^{G_1}) \oplus \chi(\nu)\tau_0(\mathbb{1}_{G_1(F)}), \\
I_{P_1}^G(\mathbb{1}_{F^\times} \otimes \chi(\det)) = J_{P_2}^G(\delta^{G_1}[1/2] \otimes \chi|\ |_F^{-1/2}) \oplus J_{P_1}^G(|\ |_F \otimes I_{B_1}^{G_1}(\chi,\chi)[-1/2]), \\
0 \longrightarrow \chi(\nu)\tau_0(\delta^{G_1}) \longrightarrow I_{P_2}^G(\delta^{G_1}[1/2] \otimes \chi|\ |_F^{-1/2}) \longrightarrow J_{P_2}^G(\delta^{G_1}[1/2] \otimes \chi|\ |_F^{-1/2}) \longrightarrow 0, \\
0 \longrightarrow \chi(\nu)\tau_0(\mathbb{1}_{G_1(F)}) \longrightarrow I_{P_2}^G(|\det|_F^{1/2} \otimes \chi|\ |_F^{-1/2}) \longrightarrow J_{P_1}^G(|\ |_F \otimes I_{B_1}^{G_1}(\chi,\chi)[-1/2]) \longrightarrow 0.
\end{gathered} \tag{4.2.3}$$

Here $\tau_0(\delta^{G_1}), \tau_0(\mathbb{1}_{G_1(F)}) \in \Pi_{\mathrm{temp}}(G(F)) \smallsetminus \Pi_{\mathrm{disc}}(G(F))$.

(4) At $\boldsymbol{\chi_\lambda} = \chi_1|\ |_F^{\lambda} \otimes |\ |_F \otimes \chi|\ |_F^{-(1+\lambda)/2} \in \mathfrak{r}_{\alpha_2}$*,* $(\lambda \geq 1)$ *or* $r_1(\boldsymbol{\chi_\lambda})$*,* $(0 \leq \lambda < 1)$ *which is neither* $\mathfrak{r}_{\rho_B}$ *nor* $\mathfrak{r}_{\alpha_1,1/2}(F^2)$*, we have*

$$0 \longrightarrow I_{P_1}^G(\chi_1|\ |_F^{\lambda} \otimes \chi(\det)\delta^{G_1}[-\frac{\lambda}{2}]) \longrightarrow I_B^G(\boldsymbol{\chi_\lambda}) \longrightarrow I_{P_1}^G(\chi_1|\ |_F^{\lambda} \otimes \chi(\det)[-\frac{\lambda}{2}]) \longrightarrow 0.$$

The representations on the right and left are both irreducible.

(5) At $\boldsymbol{\chi_\lambda} = \chi_1|\ |_F^{\lambda+1/2} \otimes \chi_1|\ |_F^{\lambda-1/2} \otimes \chi|\ |_F^{-\lambda} \in \mathfrak{r}_{\alpha_1}$*,* $(\lambda \geq 1/2)$ *or* $r_2(\boldsymbol{\chi_\lambda})$*,* $(0 \leq \lambda < 1/2)$ *which is neither* $\mathfrak{r}_{\rho_B}$ *nor* $\mathfrak{r}_{\alpha_1,1/2}(E)$*, we have*

$$0 \longrightarrow I_{P_2}^G(\chi_1(\det)\delta^{G_1}[\lambda] \otimes \chi[-\lambda]) \longrightarrow I_B^G(\boldsymbol{\chi_\lambda}) \longrightarrow I_{P_2}^G(\chi_1(\det)[\lambda] \otimes \chi[-\lambda]) \longrightarrow 0.$$

The representations on the right and left are irreducible.

(6) $I_B^G(\boldsymbol{\chi_\lambda})$ *is irreducible at any* $\boldsymbol{\chi_\lambda}$ *which is not of the form (1)–(5) above.*

Proof. The assertions (4), (5) except for the cases $\boldsymbol{\chi_\lambda} = \mathfrak{r}_{\alpha_1,0}(E) = \omega_{E/F}|\ |_F^{1/2} \otimes \omega_{E/F}|\ |_F^{1/2} \otimes \chi|\ |_F^{-1}$, $\mathfrak{r}_{\alpha_2,1} = |\ |_F \otimes |\ |_F \otimes \chi|\ |_F^{-1}$, and (6) are just the restatements of Lem.4.1 and Prop.s 3.3, 3.4, respectively.

The general theory Prop.1.7 (iii) tells us that the length of each $I_B^G(\boldsymbol{\chi_\lambda})$ in (1)–(4) of the theorem is at most $|W| = 8$. But as is stated, we can construct only 4 irreducible constituents in all the cases. Thus the principal task is to estimate the length, which exhibits the following three patterns. We often use the following consequences of Prop.1.7 (ii):

- Any irreducible constituent π of $I_B^G(\boldsymbol{\chi_\lambda})$ satisfies $\pi_B \neq 0$;
- If such π_B is irreducible, then so is π.

We also write $\pi \sqsupset \tau$ if the element $\pi - \tau \in K\mathcal{R}(G(F))$ comes from a true representation of $G(F)$.

The proof of (1), (2)—Jacquet modules. To prove (1), twisting by the character $\chi(\nu)^{-1}$, we may assume $\chi = \mathbb{1}_{F^\times}$. We need the following formulae for Jacquet modules of the representations in (4.2.0).

$$\begin{aligned}
r_B^G(I_{P_1}^G(|\ |_F^2 \otimes \delta^{G_1}[-1])) =&(2,1) + (1,2) + (1,-2) + (-2,1), && \text{(4.2.1a)}\\
r_B^G(I_{P_1}^G(|\ |_F^2 \otimes |\det|_F^{-1})) =&(2,-1) + (-1,2) + (-1,-2) + (-2,-1), && \text{(4.2.1b)}\\
r_B^G(I_{P_2}^G(\delta^{G_1}[3/2] \otimes |\ |_F^{-3/2})) =&(2,1) + (2,-1) + (-1,2) + (-1,-2), && \text{(4.2.1c)}\\
r_B^G(I_{P_2}^G(|\det|_F^{3/2} \otimes |\ |_F^{-3/2})) =&(1,2) + (1,-2) + (-2,1) + (-2,-1), && \text{(4.2.1d)}\\
r_{P_1}^G(I_{P_1}^G(|\ |_F^2 \otimes \delta^{G_1}[-1])) =&|\ |_F^2 \otimes \delta^{G_1}[-1] + |\ |_F^{-2} \otimes \delta^{G_1}[1]\\
&+ |\ |_F \otimes I_{B_1}^{G_1}(|\ |_F^{1/2} \otimes |\ |_F^{-3/2}), && \text{(4.2.1e)}\\
r_{P_2}^G(I_{P_2}^G(|\det|_F^{3/2} \otimes |\ |_F^{-3/2})) =&|\det|_F^{3/2} \otimes |\ |_F^{-3/2} + |\det|_F^{-3/2} \otimes |\ |_F^{3/2}\\
&+ I_{B_1}^{G_1}(|\ |_F \otimes |\ |_F^{-2}) \otimes |\ |_F^{1/2}. && \text{(4.2.1f)}
\end{aligned}$$

Here, we abbreviate $\boldsymbol{\chi_\lambda}$ as $\boldsymbol{\lambda}$ since $\boldsymbol{\chi}$ is trivial. Let δ_0^G be the intersection of the submodules $I_{P_1}^G(|\ |_F^2 \otimes \delta^{G_1}[-1])$ and $I_{P_2}^G(\delta^{G_1}[3/2] \otimes |\ |_F^{-3/2})$ in $I_B^G(\mathfrak{r}_{\rho_B})$. Then (4.2.1a) and (4.2.1c) implies $(\delta_0^G)_B = (2,1)$, so that $\delta_0^G \in \Pi_{\mathrm{disc}}(G(F))$ by Prop.1.10. Also the Langlands quotient

$$J_B^G(\mathfrak{r}_{\rho_B}) = \begin{cases} \mathrm{im} J_{\bar{B}|r_2(B)}(\mathfrak{r}_{\rho_B}) \circ J_{r_2(B)|B}(\mathfrak{r}_{\rho_B}) = J_{\bar{B}|r_2(B)}(\mathfrak{r}_{\rho_B})\big(I_{P_1}^G(|\ |_F^2 \otimes |\det|_F^{-1})\big), \\ \mathrm{im} J_{\bar{B}|r_1(B)}(\mathfrak{r}_{\rho_B}) \circ J_{r_1(B)|B}(\mathfrak{r}_{\rho_B}) = J_{\bar{B}|r_1(B)}(\mathfrak{r}_{\rho_B})\big(I_{P_2}^G(|\det|_F^{3/2} \otimes |\ |_F^{-3/2})\big) \end{cases}$$

is a common quotient of $I_{P_1}^G(|\ |_F^2 \otimes |\det|_F^{-1})$ and $I_{P_2}^G(|\det|_F^{3/2} \otimes |\ |_F^{-3/2})$ and hence $J_B^G(\mathfrak{r}_{\rho_B})_B = (-2,-1)$.

Now the Langlands classification (Th.1.11) asserts that *the number of irreducible constituents of* $I_P^G(\pi)$ *with* $\pi \in \mathrm{Irr}_0(M(F))$ *is at most the number of standard modules having the cuspidal support* $[M,\pi]$. By Table 1, we know that the non-tempered standard representations of $G(F)$ with the cuspidal support $[T, \mathfrak{r}_{\rho_B}]$ are

$$I_{P_1}^G(|\ |_F^2 \otimes \delta^{G_1}[-1]), \quad I_{P_2}^G(\delta^{G_1}[\tfrac{3}{2}] \otimes |\ |_F^{-3/2}), \quad I_B^G(\mathfrak{r}_{\rho_B}).$$

Thus the non-tempered irreducible subquotients of $I^G_B(\mathfrak{r}_{\rho_B})$ other than $J^G_B(\mathfrak{r}_{\rho_B})$ are the Langlands quotients $J^G_{P_1}(|\ |^2_F \otimes \delta^{G_1}[-1])$, $J^G_{P_2}(\delta^{G_1}[3/2] \otimes |\ |^{-3/2}_F)$. This together with Prop.1.10 imply

$$\begin{aligned} r^G_B(J^G_{P_1}(|\ |^2_F \otimes \delta^{G_1}[-1])) &\sqsupset (1,-2) + (-2,1), \\ r^G_B(J^G_{P_2}(\delta^{G_1}[3/2] \otimes |\ |^{-3/2}_F)) &\sqsupset (-1,2) + (-1,-2). \end{aligned}$$

But since the Jacquet modules of the first two terms on the right hand side of (4.2.1e) along B are $(\pm 2, 1)$, this forces that

$$r^G_{P_1}(J^G_{P_1}(|\ |^2_F \otimes \delta^{G_1}[-1])) \sqsupset |\ |_F \otimes I^{G_1}_{B_1}(|\ |^{1/2}_F \otimes |\ |^{-3/2}_F)$$

and hence

$$r^G_B(J^G_{P_1}(|\ |^2_F \otimes \delta^{G_1}[-1])) = r^{M_1}_{B\cap M_1} \circ r^G_{P_1}(J^G_{P_1}(|\ |^2_F \otimes \delta^{G_1}[-1])) \sqsupset (1,2) + (1,-2) + (-2,1).$$

This proves the first exact sequence in (4.2.1). The same argument using (4.2.1f) proves the second row. The rest two sequences follows from the uniqueness of the composition factors. One can easily verify that $J^G_B(\mathfrak{r}_{\rho_B}) = \mathbb{1}_{G(F)}$.

Next comes (2). We abbreviate $\omega_{E/F}$ as ω. We have the following formulae for Jacquet modules.

$$\begin{aligned} r^G_B(I^G_{P_2}(\omega(\det)\delta^{G_1}[\frac{1}{2}] \otimes \chi|\ |^{-1/2}_F) &= \big(\omega|\ |_F \otimes \omega \otimes \chi|\ |^{-1/2}_F + \omega|\ |_F \otimes \omega \otimes \omega\chi|\ |^{-1/2}_F\big) \\ &\quad + \big(\omega \otimes \omega|\ |_F \otimes \omega\chi|\ |^{-1/2}_F + \omega \otimes \omega|\ |^{-1}_F \otimes \chi|\ |^{1/2}_F\big) \end{aligned} \tag{4.2.2a}$$

$$\begin{aligned} r^G_B(I^G_{P_2}(\omega(\det)[\frac{1}{2}] \otimes \chi|\ |^{-1/2}_F) &= \big(\omega \otimes \omega|\ |_F \otimes \chi|\ |^{-1/2}_F + \omega \otimes \omega|\ |^{-1}_F \otimes \omega\chi|\ |^{1/2}_F\big) \\ &\quad + \big(\omega|\ |^{-1}_F \otimes \omega \otimes \omega\chi|\ |^{1/2}_F + \omega|\ |^{-1}_F \otimes \omega \otimes \chi|\ |^{1/2}_F\big) \end{aligned} \tag{4.2.2b}$$

$$\begin{aligned} r^G_{P_1}(I^G_{P_1}(\omega|\ |_F \otimes I^G_{B_1}(\omega\chi \otimes \chi)[-\frac{1}{2}]) &= \omega|\ |_F \otimes I^{G_1}_{B_1}(\omega\chi \otimes \chi)[-\frac{1}{2}] \\ &\quad + \omega|\ |^{-1}_F \otimes I^{G_1}_{B_1}(\omega\chi \otimes \chi)[\frac{1}{2}] + \omega \otimes I^{G_1}_{B_1}(\omega\chi|\ |^{1/2}_F \otimes \chi|\ |^{-1/2}_F) \\ &\quad + \omega \otimes I^{G_1}_{B_1}(\chi|\ |^{1/2}_F \otimes \omega\chi|\ |^{-1/2}_F). \end{aligned} \tag{4.2.2c}$$

Also note that $I^G_B(\mathfrak{r}_{\alpha_1,1/2}(E))$ share the composition factors with the central component of

$$I^G_{P_2}(\omega(\det)\delta^{G_1}[\frac{1}{2}] \otimes \omega\chi|\ |^{-1/2}_F) \hookrightarrow I^G_B(r_2(\mathfrak{r}_{\alpha_1,1/2}(E))) \twoheadrightarrow I^G_{P_2}(\omega(\det)[\frac{1}{2}] \otimes \omega\chi|\ |^{-1/2}_F).$$

The Jacquet modules of the representations in this sequence are given by the formulae (4.2.2a–c) with χ replaced by $\omega\chi$.

Now we proceed as in the proof of (1). One can verify that the intersection $\chi(\nu)\delta^G_0(E)$ of $I^G_{P_2}(\omega(\det)\delta^{G_1}[1/2] \otimes \chi|\ |^{-1/2}_F)$, $I^G_{P_2}(\omega(\det)\delta^{G_1}[1/2] \otimes \omega\chi|\ |^{-1/2}_F)$ and the common quotient $J^G_{P_1}(\omega|\ |_F \otimes I^{G_1}_{B_1}(\omega\chi, \chi)[-1/2])$ of $I^G_{P_2}(\omega(\det)[1/2] \otimes \chi|\ |^{-1/2}_F)$, $I^G_{P_2}(\omega(\det)[1/2] \otimes \omega\chi|\ |^{-1/2}_F)$ satisfy

$$r^G_B(\chi(\nu)\delta^G_0(E)) \sqsubset \omega|\ |_F \otimes \omega \otimes \chi|\ |^{-1/2}_F + \omega|\ |_F \otimes \omega \otimes \omega\chi|\ |^{-1/2}_F, \tag{4.2.2d}$$

$$r^G_B(J^G_{P_1}(\omega|\ |_F \otimes I^{G_1}_{B_1}(\omega\chi \otimes \chi)[-1/2])) \sqsubset \omega|\ |^{-1}_F \otimes \omega \otimes \chi|\ |^{1/2}_F + \omega|\ |^{-1}_F \otimes \omega \otimes \omega\chi|\ |^{1/2}_F, \tag{4.2.2e}$$

respectively. Then (4.2.2c) and

$$r^{M_1}_{B\cap M_1}(\chi_1\otimes I^{G_1}_{B_1}(\chi_2\otimes\chi_3)) = \chi_1\otimes\chi_2/\chi_3\otimes\chi_3+\chi_1\otimes\chi_3/\chi_2\otimes\chi_2$$

show that

$$r^G_{P_1}(\chi(\nu)\delta^G_0(E)) = \omega|\,|_F\otimes I^{G_1}_{B_1}(\omega\chi\otimes\chi)[-\frac{1}{2}],$$

$$r^G_{P_1}(J^G_{P_1}(\omega|\,|_F\otimes I^{G_1}_{B_1}(\omega\chi\otimes\chi)[-\frac{1}{2}]) = \omega|\,|_F^{-1}\otimes I^{G_1}_{B_1}(\omega\chi\otimes\chi)[\frac{1}{2}].$$

Thus $\chi(\nu)\delta_0(E)$ is irreducible and (4.2.2d, e) become equalities. These in turn imply that

$$r^G_B(J^G_{P_2}(\omega(\det)\delta^{G_1}[\frac{1}{2}]\otimes\chi|\,|_F^{-1/2})\sqsubset\omega\otimes\omega|\,|_F\otimes\omega\chi|\,|_F^{-1/2}+\omega\otimes\omega|\,|_F^{-1}\otimes\chi|\,|_F^{1/2},$$

$$r^G_B(I^G_{P_2}(\omega(\det)\delta^{G_1}[\frac{1}{2}]\otimes\omega\chi|\,|_F^{-1/2})\sqsubset\omega\otimes\omega|\,|_F\otimes\chi|\,|_F^{-1/2}+\omega\otimes\omega|\,|_F^{-1}\otimes\omega\chi|\,|_F^{1/2}.$$

Again (4.2.2c) shows that these are actually equalities, and the first sequence in (4.2.2) is proved. The second row can now be easily verified by the uniqueness of the composition factors.

The proof of (3)—commuting algebra. We may and do assume that $\chi=\mathbb{1}_{F^\times}$. We have the following formulae for Jacquet modules:

$$r^G_B(I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\delta^{G_1})) = 2(0,1)+2(1,0),$$

$$r^G_B(I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\mathbb{1}_{G_1(F)})) = 2(0,-1)+2(-1,0),$$

$$r^G_B(I^G_{P_2}(\delta^{G_1}[\frac{1}{2}]\otimes|\,|_F^{-1/2})) = 2(1,0)+(0,1)+(0,-1),$$

$$r^G_B(I^G_{P_2}(|\det|_F^{1/2}\otimes|\,|_F^{-1/2})) = 2(-1,0)+(0,1)+(0,-1).$$

First notice that $I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\delta^{G_1})$ and $I^G_{P_2}(|\det|_F^{1/2}\otimes|\,|_F^{-1/2})$ (resp. $I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\mathbb{1}_{G_1(F)})$ and $I^G_{P_2}(\delta^{G_1}[1/2]\otimes|\,|_F^{-1/2})$) must share an irreducible constituent, which we denote by $\tau_0(\mathbb{1}_{G_1(F)})$ (resp. J). Otherwise, we must have $r^G_B(I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\delta^{G_1})\sqsubset r^G_B(I^G_{P_2}(\delta^{G_1}[1/2]\otimes|\,|_F^{-1/2}))$ but this is impossible. These $\tau_0(\mathbb{1}_{G_1(F)}), J$ are irreducible since so are $r^G_B(\tau_0(\mathbb{1}_{G_1(F)}))=(0,1)$, $r^G_B(J)=(0,-1)$. We write $\tau_0(\delta^{G_1})$ for the orthogonal complement of $\tau_0(\mathbb{1}_{G_1(F)})$ in $I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\delta^{G_1})$, so that $r^G_B(\tau_0(\delta^{G_1}))=2(1,0)+(0,1)$. It follows from Langlands-Casselman's criterion that $\tau_0(\delta^{G_1}),\tau_0(\mathbb{1}_{G_1(F)})$ are tempered but not square integrable.

On the other hand, (4.1) gives

$$\dim\mathrm{End}_{G(F)}(I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\delta^{G_1})) = \dim\mathrm{End}_{G(F)}(I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\mathbb{1}_{G_1(F)}))\le 2,$$

and hence the length of $I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\chi(\det)\delta^{G_1})$ and $I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\chi(\det))$ are at most two. This immediately proves the first row in (4.2.3), namely $\tau_0(\delta^{G_1})$ is irreducible. Since $\tau_0(\delta^{G_1})$ is not a subquotient of $I^G_{P_2}(|\det|_F^{1/2}\otimes|\,|_F^{-1/2})$, it must be in the composition series of $I^G_{P_2}(\delta^{G_1}[1/2]\otimes|\,|_F^{-1/2})$. Granting $r^G_B(J), r^G_B(\tau_0(\delta^{G_1}))$, we deduce the third row in (4.2.3) with $J\simeq J^G_{P_2}(\delta^{G_1}[1/2]\otimes|\,|_F^{-1/2})$. Similarly, the orthogonal complement $J^\perp$ of $J^G_{P_2}(\delta^{G_1}[1/2]\otimes|\,|_F^{-1/2})$ in $I^G_{P_1}(\mathbb{1}_{F^\times}\otimes\mathbb{1}_{G_1(F)})$ must be a composition factor of $I^G_{P_2}(|\det|_F^{1/2}\otimes|\,|_F^{-1/2})$. Again looking at $r^G_B(J^\perp)=2(-1,0)+(0,-1)$, we obtain the last exact sequence of (4.2.3), where $J^\perp$ must be the unique non-tempered constituent $J^G_{P_1}(|\,|_F\otimes I^{G_1}_{B_1}(\mathbb{1}_{T_1(F)})[-1/2])$.

Proof of (4), (5) in the case $\boldsymbol{\chi}_{\boldsymbol{\lambda}} = \mathfrak{r}_{\alpha_2,1} = |\ |_F \otimes |\ |_F \otimes \chi[-1]$—spherical functions. The proof of (5) at $\boldsymbol{\chi}_{\boldsymbol{\lambda}} = \mathfrak{r}_{\alpha_1,0}(E)$ is similar to that of (2). On the other hand, the proof of (4), (5) at $\mathfrak{r}_{\alpha_2,1}$ depends on computation of spherical functions [Rod88, §5].

For $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathbb{C}^2$, we have the unramified quasi-character $e^{\boldsymbol{\lambda}} = |\ |_F^{\lambda_1} \otimes |\ |_F^{\lambda_2} \otimes |\ |_F^{-\lambda_+}$ as in §1.5 and the associated unramified principal series representation $I(\boldsymbol{\lambda}) := I_B^G(e^{\boldsymbol{\lambda}})$. This contains a $\mathbf{K}$-invariant vector

$$\phi_{\boldsymbol{\lambda}}^{\mathbf{K}}(utk) = t^{\boldsymbol{\lambda}+\rho_B}, \quad u \in U_0(F), t \in T(F), k \in \mathbf{K},$$

where we have written $t^{\boldsymbol{\lambda}} := e^{\boldsymbol{\lambda}}(t)$, $\rho_B = (2,1)$. The matrix coefficient

$$\Xi_{\boldsymbol{\lambda}}(g) := \langle I(\boldsymbol{\lambda}, g)\phi_{\boldsymbol{\lambda}}^{\mathbf{K}}, \phi_{-\boldsymbol{\lambda}}^{\mathbf{K}}\rangle = \int_{\mathbf{K}} \phi_{\boldsymbol{\lambda}}^{\mathbf{K}}(kg)\, dk$$

is called the *spherical function* of parameter $\boldsymbol{\lambda}$. One can compute $\Xi_{\boldsymbol{\lambda}}$ under the assumption that $\boldsymbol{\lambda}$ is regular: $\lambda_1 \pm \lambda_2 \neq 0$, $\lambda_1, \lambda_2 \neq 0$. In fact, by a detailed analysis of the Bruhat filtration applied to $\phi_{\boldsymbol{\lambda}}^{\mathbf{K}}$, Macdonald obtained the formula [Cas80, Th.4.2]:

$$\begin{aligned}\Xi_{\boldsymbol{\lambda}}(t) =&\mathrm{meas}(\mathbf{I}\widetilde{w}_0\mathbf{I}) \sum_{w \in W} \left(\prod_{\alpha \in \Sigma_B} \frac{\zeta(-\alpha^\vee(w(\boldsymbol{\lambda})))}{\zeta(1-\alpha^\vee(w(\boldsymbol{\lambda})))} \right) e^{w(\boldsymbol{\lambda})+\rho_B} \\ =&\mathrm{meas}(\mathbf{I}\widetilde{w}_0\mathbf{I}) \sum_{w \in W} \left(\prod_{\alpha \in \Sigma_B} \frac{1-q_F^{1-\alpha^\vee(w(\boldsymbol{\lambda}))}}{1-q_F^{-\alpha^\vee(w(\boldsymbol{\lambda}))}} \right) e^{w(\boldsymbol{\lambda})+\rho_B}.\end{aligned} \tag{4.2.4a}$$

Here, $\mathbf{I}$ denotes the standard *Iwahori subgroup* of $G(F)$ and $\zeta(s)$ is the Dedekind zeta factor of F. The formula is valid on the subset $\{t \in T(F) \,|\, |\alpha^\vee(t)|_F \leq 1,\ \alpha \in \Sigma_B\}$. This actually extends to a holomorphic function of $\boldsymbol{\lambda} \in \mathbb{C}^2$, although it is not obvious from (4.2.4a). At a point $\lambda\mathfrak{r}_{\alpha_2,1} = (\lambda, \lambda)$, this becomes (after a lengthy calculation of logarithmic derivatives) [Rod88, Prop.9]:

$$\Xi_{\boldsymbol{\lambda}}(t) = \mathrm{meas}(\mathbf{I}\widetilde{w}_0\mathbf{I}) \sum_{\substack{w \in W \\ w(\alpha_1) \in \Sigma_B}} B(w, \boldsymbol{\lambda}) A(w, t, \boldsymbol{\lambda}) t^{w(\boldsymbol{\lambda})+\rho_B}$$

where

$$\begin{aligned}B(w, \boldsymbol{\lambda}) :=& \operatorname{sgn}(w) q_F^{\langle \rho_{\widehat{B}} - w^{-1}(\rho_{\widehat{B}}), \boldsymbol{\lambda}\rangle} \prod_{\alpha \in \Sigma_B \smallsetminus \{\alpha_1\}} \zeta(-\alpha^\vee(\boldsymbol{\lambda})), \\ A(w, t, \boldsymbol{\lambda}) :=& \Big(\langle \rho_{\widehat{B}}, w(\alpha_1)\rangle + \mathrm{val}_F(t^{w(\alpha_1)}) \Big) \prod_{\alpha \in \Sigma_B} \zeta(1-\alpha^\vee(w(\boldsymbol{\lambda})))^{-1} \\ &+ q_F^{-1} \sum_{\alpha \in \Sigma_B} \langle \alpha^\vee, w(\alpha_1)\rangle q_F^{\langle \alpha^\vee, w(\boldsymbol{\lambda})\rangle} \prod_{\beta \in \Sigma_B \smallsetminus \{\alpha\}} \zeta(1-\beta^\vee(w(\boldsymbol{\lambda})))^{-1}.\end{aligned}$$

We now specialize this to $\mathfrak{r}_{\alpha_2,1} = (1,1)$. Note that the factor $B(w, \boldsymbol{\lambda})$ is always non-zero. The set $\{w \in W \,|\, w(\alpha_1) \in \Sigma_B\}$ consists of $1, r_2, r_1 r_2, w_{M_2}$.

(i) For $w = 1$, we have $w(\boldsymbol{\lambda}) = (1,1)$ and $\alpha_2^\vee(1,1) = \beta_2^\vee(1,1) = 1$, where $\beta_2 := 2\alpha_1 + \alpha_2$ is the highest root in Σ_B. Thus $\zeta(1-\alpha^\vee(w(\boldsymbol{\lambda})))^{-1} = 1 - q_F^{\alpha^\vee(w(\boldsymbol{\lambda}))-1} = 0$ for $\alpha = \alpha_2, \beta_2$, and hence both the first and the second term of $A(1, t, \boldsymbol{\lambda})$ is zero.

(ii) For $w = r_2$, $\alpha(r_2(\boldsymbol{\lambda})) = \alpha(1,-1) = 1$ if and only if $\alpha = \beta_2$. Thus the first row of $A(r_2, t, \boldsymbol{\lambda})$ equals zero, while the second row becomes

$$\langle \beta_2^\vee, r_2(\alpha_1)\rangle \prod_{\beta\neq\beta_2} \zeta(1-\alpha^\vee(w(\boldsymbol{\lambda})))^{-1} \neq 0.$$

(iii) For $w = r_1 r_2$, $\alpha^\vee(r_1 r_2(\boldsymbol{\lambda})) = \alpha^\vee(-1,1) = 1$ only for $\alpha = \alpha_2$. Thus again the first row of $A(r_1 r_2, t, \boldsymbol{\lambda})$ is zero but the second row reads

$$\langle \alpha_2^\vee, \beta_1\rangle \prod_{\beta\neq\alpha_2} \zeta(1-\alpha^\vee(w(\boldsymbol{\lambda})))^{-1} \neq 0.$$

(iv) For $w = w_{M_2}$, we have no $\alpha \in \Sigma_B$ with $\alpha^\vee(w_{M_2}(\boldsymbol{\lambda})) = 1$, so that both the first and second rows of $A(w_{M_2}, t, \boldsymbol{\lambda})$ are non-zero.

These show that $\Xi_{\mathfrak{t}_{\alpha_1,1}}(t)$ is a linear combination of characters $t^{(1,-1)+\rho_B}$, $t^{(-1,1)+\rho_B}$, $t^{(-1,-1)+\rho_B}$ and the log term $t^{(-1,-1)+\rho_B}\mathrm{val}_F(t^{\alpha_1})$ with non-zero coefficients. Let π be the unique $\mathbf{K}$-spherical irreducible subquotient of $I(\mathfrak{t}_{\alpha_1,1})$. Let $\phi^{\mathbf{K}}_{\mathfrak{t}_{\alpha_1,1}}$, $\phi^{\mathbf{K}}_{-\mathfrak{t}_{\alpha_1,1}}$ be spherical vectors in π, $\pi^\vee$ satisfying $\langle \phi^{\mathbf{K}}_{\mathfrak{t}_{\alpha_1,1}}, \phi^{\mathbf{K}}_{-\mathfrak{t}_{\alpha_1,1}}\rangle = 1$, so that $\Xi_{\mathfrak{t}_{\alpha_1,1}}(g) = \langle \pi(g)\phi^{\mathbf{K}}_{\mathfrak{t}_{\alpha_1,1}}, \phi^{\mathbf{K}}_{-\mathfrak{t}_{\alpha_1,1}}\rangle$. For $t \in T(F)$ with $\alpha^\vee(t) \ll 1$, $(\forall\alpha \in \Sigma_B)$, $t^{-\rho_B}\Xi_{\mathfrak{t}_{\alpha_1,1}}(t)$ is equal to its *constant term* along B:

$$\Xi_{\mathfrak{t}_{\alpha_1,1},B}(t) = \langle \pi_B(t) j_B(\phi^{\mathbf{K}}_{\mathfrak{t}_{\alpha_1,1}}), j_{\bar{B}}(\phi^{\mathbf{K}}_{-\mathfrak{t}_{\alpha_1,1}})\rangle.$$

Combining this with the above calculation of $\Xi_{\mathfrak{t}_{\alpha_1,1}}(t)$, we see that

$$r^G_B(\pi) = (1,-1) + (-1,1) + 2(-1,-1).$$

Using this and the formulas

$$\begin{aligned} r^G_B(I^G_{P_1}(|\ |_F \otimes \delta^{G_1}[-\tfrac{1}{2}])) &= 2(1,1) + (1,-1) + (-1,1), \\ r^G_B(I^G_{P_1}(|\ |_F \otimes |\det|_F^{-1/2})) &= (1,-1) + (-1,1) + 2(-1,-1), \end{aligned}$$

one can show that both $I^G_{P_1}(|\ |_F \otimes \delta^{G_1}[-1/2])$ and $I^G_{P_1}(|\ |_F \otimes |\det|_F^{-1/2})$ are irreducible. The irreducibility of $I^G_{P_1}(|\ |_F \otimes \delta^{G_1}[-1/2])$ follows from the fact that it is the correspondent of $I^G_{P_1}(|\ |_F \otimes |\det|_F^{-1/2})$ under the Zelevinsky-Aubert-Schneider-Stuhler (ZASS for short) involution (up to sign) and that the involution sends irreducibles to irreducibles (up to sign). As for these facts as well as the elementary definition[4] of the ZASS involution, see [Aub95]. □

4.3 Irreducible representations supported on M_1

We now classify the irreducible admissible representations of $G(F)$ whose cuspidal support is of the form $[M_1, \sigma]$. We write $\sigma = \chi|\ |_F^\lambda \otimes \pi[-\lambda/2] \in \mathrm{Irr}_0(M_1(F))$ as in Prob.3.1.

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[4] The definition given in [Aub95] is an analogue of the *Curtis-Kawanaka-Alvis duality* on the Grothendieck group of finite dimensional representations of finite reductive groups.

Theorem 4.3 ([Sha90] Rem.8.5)**.** *(i)* $I_{P_1}^G(\chi|\;|_F^\lambda\otimes\pi[-\lambda/2])$, $\chi\in\Pi(F^\times)$, $\pi\in\Pi_0(G_1(F))$ *is irreducible at any* $\lambda\geq 0$ *except if* $\chi^2=\mathbb{1}_{F^\times}$ *and* $\chi(\det)\pi\simeq\pi$.
(ii) If $\chi=\mathbb{1}_{F^\times}$*, then* $I_{P_1}^G(\chi\otimes\pi)\simeq\tau_1(\pi)^+\oplus\tau_1(\pi)^-$ *for* $\tau_1(\pi)^\pm\in\Pi_{\rm temp}(G(F))\smallsetminus\Pi_{\rm disc}(G(F))$. $I_{P_1}^G(\chi|\;|_F^\lambda\otimes\pi[-\lambda/2])$ *is irreducible at any* $\lambda>0$.
(iii) If $\chi=\omega_{E/F}$ *for some quadratic extension* E/F*, then* $\omega_{E/F}(\det)\pi\simeq\pi$ *if and only if* $\pi=\pi(\omega)$ *(*§2.3*) for some* $\omega\in\Pi(E^\times)$ *([Lan80, Lem.7.17]). Then* $I_{P_1}^G(\chi|\;|_F^\lambda\otimes\pi(\omega)[-\lambda/2])$ *is irreducible except at* $\lambda=1$*, where we have*

$$0\longrightarrow\delta_1(E,\omega)\longrightarrow I_{P_1}^G(\chi|\;|_F\otimes\pi[-\frac{1}{2}])\longrightarrow J_{P_1}^G(\chi|\;|_F\otimes\pi[-\frac{1}{2}])\longrightarrow 0,$$

with $\delta_1(E,\omega)\in\Pi_{\rm disc}(G(F))$.

Proof. Since $w_{M_1}(m_1(a,g))=m_1(a^{-1}\det g,g)$, $W(M_1)_{\chi\otimes\pi}$ is non-trivial if and only if $\chi^2=\mathbb{1}_{F^\times}$ and $\chi(\det)\pi\simeq\pi$. Thus (i) follows from Prop.1.12 (i).

We now need two consequences of the Langlands-Shahidi theory. In [Sha90, §7], Shahidi defined a large class of L and ε-factors. Writing $\rho^\vee_{\bar P_1|P_1}$ for the adjoint representation of ${}^LM_1$ on the Lie algebra $\widehat{\mathfrak{u}}_1$ of $\widehat{U}_1$ (see §3.1), the factors $L(s,\chi\otimes\pi,\rho^\vee_{\bar P_1|P_1})$ and $\varepsilon(s,\chi\otimes\pi,\rho^\vee_{\bar P_1|P_1},\psi)$ are among the list of factors. His local factors are defined from the action of the intertwining operator on the space of Whittaker functionals. One of the advantages of this method is a direct relationship between the Plancherel measures and γ-factors [Sha90, Cor.3.6]. For example, in the present case, we have

$$\begin{aligned}\mu(\chi|\;|_F^\lambda\otimes\pi[-\lambda/2])=\gamma_{M_1}&\frac{\varepsilon(\lambda,\chi\otimes\pi,\rho^\vee_{\bar P_1|P_1},\bar\psi)L(1-\lambda,\chi^{-1}\otimes\pi^\vee,\rho^\vee_{\bar P_1|P_1})}{L(\lambda,\chi\otimes\pi,\rho^\vee_{\bar P_1|P_1})}\\ &\times\frac{\varepsilon(-\lambda,\chi^{-1}\otimes\pi^\vee,\rho^\vee_{\bar P_1|P_1},\psi)L(1+\lambda,\chi\otimes\pi,\rho^\vee_{\bar P_1|P_1})}{L(-\lambda,\chi^{-1}\otimes\pi^\vee,\rho^\vee_{\bar P_1|P_1})}.\end{aligned}$$

Moreover, these γ-factors also fit into the global functional equations such as [*loc.cit.*, Th.3.5 (4)]:

$$\begin{aligned}L^S(s,{\rm Ad}^2(\pi_{\mathbb{A}})\times\chi_{\mathbb{A}})=&\prod_{v\in S}\frac{\varepsilon(s,\chi_v\otimes\pi_v,\rho^\vee_{\bar P_1|P_1},\psi_v)L(1-s,\chi_v^{-1}\otimes\pi_v^\vee,\rho^\vee_{\bar P_1|P_1})}{L(s,\chi_v\otimes\pi_v,\rho^\vee_{\bar P_1|P_1})}\\ &\times L^S(1-s,{\rm Ad}^2(\pi_{\mathbb{A}}^\vee)\times\chi_{\mathbb{A}}^{-1}).\end{aligned}$$

Here $\pi_{\mathbb{A}}$ is an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$ and $\chi_{\mathbb{A}}$ is an idele class character. $L^S(s,{\rm Ad}^2(\pi_{\mathbb{A}})\times\chi_{\mathbb{A}})$ denotes the *adjoint square L-function* of $\pi_{\mathbb{A}}$ twisted by $\chi_{\mathbb{A}}$ [GJ78]. It follows from the uniqueness of the γ-factor [*loc.cit.*, Th.3.5] that

$$L(s,\chi\otimes\pi,\rho^\vee_{\bar P_1|P_1})=L(s,{\rm Ad}^2(\pi)\times\chi).$$

From these two results, we conclude that the poles (resp. the zeros) of $\mu(\chi|\;|_F^\lambda\otimes\pi[-\lambda/2])$ in the region $\lambda>0$ (resp. at $\lambda=0$) come from those of

$$L(1-\lambda,{\rm Ad}^2(\pi^\vee)\times\chi^{-1})\quad(\text{resp. }L(\lambda,{\rm Ad}^2(\pi)\times\chi)L(-\lambda,{\rm Ad}^2(\pi^\vee)\times\chi^{-1})).$$

If π is extraordinary, $L(s, \mathrm{Ad}^2(\pi) \times \chi) = 1$ [GJ78, Prop.3.4] and $\mu(\chi|\ |_F^{\lambda} \otimes \pi[-\lambda/2])$ has neither poles nor zeros. Suppose π is dihedral: $\pi \simeq \pi(\omega)$ for a quadratic extension E/F and $\omega \in \Pi(E^\times)$. Then it is shown in [*loc.cit.*, pp.488-489] that

$$L(s, \mathrm{Ad}^2(\pi) \times \chi) = L_E(s, \omega\sigma(\omega)^{-1}\chi(\mathrm{N}_{E/F}))L(s, \chi\omega_{E/F}).$$

Here σ denotes the generator of the Galois group $\mathrm{Gal}(E/F)$. The second factor on the right has a simple pole at $s = 0$ if and only if $\chi = \omega_{E/F}$. In this case since $\pi(\omega)$ is cuspidal: $\omega\sigma(\omega)^{-1} \neq \mathbb{1}_{E^\times}$, $s = 0$ is a simple pole of $L(s, \mathrm{Ad}^2(\pi) \times \chi)$. We still have to compute the poles of $L_E(s, \omega\sigma(\omega)^{-1}\chi(\mathrm{N}_{E/F}))$. This has a pole if and only if $\omega\sigma(\omega)^{-1} = |\ |_E^{\alpha}\chi(\mathrm{N}_{E/F})^{-1}$ for some $\alpha \in \mathbb{C}$. Applying σ, we find that $\omega\sigma(\omega)^{-1}$ is of order two. By the local classfield theory, we have a quadratic extension K/E such that $\omega\sigma(\omega)^{-1} = \omega_{K/E}$. Write E, E', E'' for the three quadratic extension of F inside the biquadratic extension K/F. By [Lan80, Lem.7.17], there exist $\omega' \in \Pi(E'^\times)$, $\omega'' \in \Pi(E''^\times)$ such that $\pi(\omega) \simeq \pi(\omega') \simeq \pi(\omega'')$. Then we must have

$$L(s, \mathrm{Ad}^2(\pi) \times \chi) = L(s, \chi\omega_{E/F})L(s, \chi\omega_{E'/F})L(s, \chi\omega_{E''/F}).$$

This shows that $L(s, \mathrm{Ad}^2(\pi) \times \chi)$ has a pole at $s = 0$ if and only if $\chi = \omega_{E/F}$ for some quadratic extension E/F and $\pi \simeq \pi(\omega)$ for some $\omega \in \Pi(E^\times)$, and has neither zeros nor poles otherwise. Having determined the poles and zeros of $\mu(\chi|\ |_F^{\lambda} \otimes \pi[-\lambda/2])$, now, the reducibility results in (ii) and (iii) follow from Prop.1.12 (2) and (3), respectively.

At the reducible points, the length of $I_{P_1}^G(\chi|\ |_F^{\lambda} \otimes \pi[-\lambda/2])$ is two by Prop.1.7 (iii). Since $I_{P_1}^G(\mathbb{1}_{F^\times} \otimes \pi)$ is unitarizable, it is completely reducible and the two irreducible summands are tempered. On the other hand, the reducible representation $I_{P_1}^G(\omega_{E/F}|\ |_F \otimes \pi(\omega)[-1/2])$ admits a unique irreducible submodule $\delta_1(E, \omega) := \ker J_{\bar{P}_1|P_1}(\omega_{E/F}|\ |_F \otimes \pi(\omega)[-1/2])$ and the irreducible quotient $J_{P_1}^G(\omega_{E/F}|\ |_F \otimes \pi(\omega)[-1/2])$. Granting $I_{P_1}^G(\omega_{E/F}|\ |_F \otimes \pi(\omega)[-1/2])_{P_1} \simeq \omega_{E/F}|\ |_F \otimes \pi(\omega)[-1/2] \oplus \omega_{E/F}|\ |_F^{-1} \otimes \pi(\omega)[1/2]$, we must have

$$J_{P_1}^G(\omega_{E/F}|\ |_F \otimes \pi(\omega)[-\frac{1}{2}])_{P_1} \simeq \omega_{E/F}|\ |_F^{-1} \otimes \pi(\omega)[\frac{1}{2}], \quad \delta_1(E, \omega)_{P_1} \simeq \omega_{E/F}|\ |_F \otimes \pi(\omega)[-\frac{1}{2}],$$

and hence $\delta_1(E, \omega)$ is square integrable by Prop.1.10. □

4.4 Irreducible representations supported on M_2

Finally we classify the irreducible admissible representations with cuspidal supports of the form $[M_2, \sigma]$. Recall the notation $\sigma = \pi[\lambda] \otimes \chi|\ |_F^{-\lambda}$ of Prob.3.1.

Theorem 4.4 ([Sha91] Prop.6.1)**.** *(i)* $I_{P_2}^G(\pi[\lambda] \otimes \chi|\ |_F^{-\lambda})$, *(*$\pi \in \Pi_0(G_1(F))$, $\chi \in \Pi(F^\times)$, $\lambda \geq 0$*) is irreducible except if* $\omega_\pi = \mathbb{1}_{F^\times}$.
(ii) If $\omega_\pi = \mathbb{1}_{F^\times}$ *(i.e.,* $\pi^\vee \simeq \pi$*),* $I_{P_2}^G(\pi[\lambda] \otimes \chi|\ |_F^{-\lambda})$ *as above is irreducible except at* $\lambda = 1/2$*, where we have*

$$0 \longrightarrow \chi(\nu)\delta_2(\pi) \longrightarrow I_{P_2}^G(\pi[\frac{1}{2}] \otimes \chi|\ |_F^{-1/2}) \longrightarrow J_{P_2}^G(\pi[\frac{1}{2}] \otimes \chi|\ |_F^{-1/2}) \longrightarrow 0,$$

with $\delta_2(\pi) \in \Pi_{\mathrm{disc}}(G(F))$.

Proof. This is proved by a similar argument as in the proof of Th.4.3. This time the Plancherel measure $\mu(\pi[\lambda]\otimes\chi|\ |_F^{-\lambda})$ is given by

$$\begin{aligned}\mu(\pi[\lambda]\otimes\chi|\ |_F^{-\lambda}) =&\gamma_{M_2}\frac{\varepsilon(\lambda,\pi,\bar{\psi})\varepsilon(2\lambda,\omega_\pi,\bar{\psi})L(1-\lambda,\pi^\vee)L(1-2\lambda,\omega_\pi^{-1})}{L(\lambda,\pi)L(2\lambda,\omega_\pi)}\\ &\times\frac{\varepsilon(-\lambda,\pi^\vee,\psi)\varepsilon(-2\lambda,\omega_\pi^{-1},\psi)L(1+\lambda,\pi)L(1+2\lambda,\omega_\pi)}{L(-\lambda,\pi^\vee)L(-2\lambda,\omega_\pi^{-1})}\\ =&\gamma_{M_2}q_F^{c(\pi)+c(\omega_\pi)+3\mathrm{ord}\psi}\frac{L(1-2\lambda,\omega_\pi^{-1})L(1+2\lambda,\omega_\pi)}{L(2\lambda,\omega_\pi)L(-2\lambda,\omega_\pi^{-1})}.\end{aligned}$$

Here $c(\pi)$ and $c(\omega_\pi)$ are *conductors* of π and ω_π, respectively (see [Cas73]), and $\mathrm{ord}\psi$ is the largest integer n such that $\psi|_{\mathfrak{p}^{-n}}$ is trivial. □

5 Unitary dual

In this final section, we give a list of unitarizable classes in $\mathrm{Irr}(G(F))\setminus\mathrm{Irr}_0(G(F))$.

Theorem 5.1 (Th.4.4 in [ST93]). *The set $\Pi(G(F))\setminus\Pi_0(G(F))$ consists of the following classes. In any case, the similitude component $\chi\in\Pi(F^\times)$ must be unitary.*
(1) The set $\Pi_{\mathrm{disc}}(G(F))\setminus\Pi_0(G(F))$ consists of the following representations.

Reference	*Representation*	*Conditions*	*Label in [RS]*
Th.4.2 (1)	$\chi(\nu)\delta_0^G$	—	*Tabel A.1 IV.a*
Th.4.2 (2)	$\chi(\nu)\delta_0^G(E)$	*E/F quad. ext.*	*Tabel A.1 V.a*
Th.4.3 (iii)	$\delta_1(E,\omega)$	*E/F quad. ext.* $\omega\in\Pi(E^\times)$, $\sigma(\omega)\neq\omega$	*Tabel A.1 IX.a*
Th.4.4 (ii)	$\chi(\nu)\delta_2(\pi)$	$\pi\in\Pi_0(G_1(F))$, $\pi^\vee\simeq\pi$	*Tabel A.1 XI.a*

(2) The set $\Pi_{\mathrm{temp}}(G(F))\setminus\Pi_{\mathrm{disc}}(G(F))$ consists of the following classes.

Reference	*Representation*	*Conditions*	*Label in [RS]*
Th.4.2 (6)	$I_B^G(\boldsymbol{\chi})$	$\boldsymbol{\chi}\in\Pi(T(F))$	*Table A.1 I*
Th.4.2 (3)	$\chi(\nu)\tau_0(\delta^{G_1})$, $\chi(\nu)\tau_0(\mathbb{1}_{G_1(F)})$	—	*Table A.1 VI.a, b*
Th.4.3 (ii)	$\tau_1(\pi)^\pm$	$\pi\in\Pi_0(G_1(F))$	*Table A.1 VIII.a, b*
Th.4.2 (4)	$I_{P_1}^G(\chi_1\otimes\chi(\det)\delta^{G_1})$	$\chi_1\neq\mathbb{1}_{F^\times},\in\Pi(F^\times)$	*Table A.1 III.a*
Th.4.3 (i), (iii)	$I_{P_1}^G(\chi_1\otimes\pi)$	$\chi_1\neq\mathbb{1}_{F^\times},\in\Pi(F^\times)$	*Table A.1 VII*
Th.4.2 (5)	$I_{P_2}^G(\chi_1(\det)\delta^{G_1}\otimes\chi)$	$\chi_1\in\Pi(F^\times)$	*Table A.1 II.a*
Th.4.4	$I_{P_2}^G(\pi\otimes\chi)$	$\pi\in\Pi_0(G_1(F))$	*Table A.1 X*

(3) $\Pi(G(F))\setminus\Pi_{\mathrm{temp}}(G(F))$ is a disjoint union of the following three subsets.
(i) The following Langlands quotients of standard modules induced from B:

Reference	*Representation*	*Conditions*	*Label in [RS]*
Th.4.2 (1)	$\chi(\nu)$	—	*Table A.2 IV.d*
Th.4.2 (6)	$I_B^G(\omega_{E/F}\|\ \|_F^{\lambda_1}\otimes\omega_{E/F}\|\ \|_F^{\lambda_2}\otimes\chi\|\ \|_F^{-\lambda_+})$	*E/F quad. ext.* $\lambda_1>\lambda_2>0$, $\lambda_1+\lambda_2<1$	*Table A.2 I*
Th.4.2 (5)	$I_{P_2}^G(\omega_{E/F}(\det)[\lambda]\otimes\chi\|\ \|_F^{-\lambda})$	*E/F quad. ext.* $0<\lambda<1/2$	*Table A.2 II.b*

In this table, we have written $\lambda_+ := (\lambda_1 + \lambda_2)/2$ and allow the case $E \simeq F^2$.
(ii) The following Langlands quotients of standard modules induced from P_1:

Reference	*Representation*	*Conditions*	*Label in [RS]*
*Th.*4.2 *(6)*	$I_{P_1}^G(\omega_{E/F}\vert\ \vert_F^\lambda \otimes \pi(\omega)[-\lambda/2])$	E/F *quad. ext.* *include* $E = F^2$ $\omega = \chi \circ \mathrm{N}_{E/F}$, $0 < \lambda < 1$	*Table A.2 I*
*Th.*4.2 *(2), (3)*	$J_{P_1}^G(\omega_{E/F}\vert\ \vert_F \otimes \pi(\omega)[-1/2])$	E/F *quad. ext.* *include* $E = F^2$ $\omega = \chi \circ \mathrm{N}_{E/F}$	*Table A.2 V.d, VI.d*
*Th.*4.3 *(iii)*	$I_{P_1}^G(\omega_{E/F}\vert\ \vert_F^\lambda \otimes \pi(\omega)[-\lambda/2])$	E/F *quad. ext.* $\omega \neq \sigma(\omega), \in \Pi(E^\times)$, $0 < \lambda < 1$	*Table A.2 VII*
*Th.*4.3 *(iii)*	$J_{P_1}^G(\omega_{E/F}\vert\ \vert_F \otimes \pi(\omega)[-1/2])$	E/F *quad. ext.* $\omega \neq \sigma(\omega), \in \Pi(E^\times)$	*Table A.2 IX.b*

(iii) The following Langlands quotients of standard induced representations from P_2:

Reference	*Representation*	*Conditions*	*Label in [RS]*
*Th.*4.2 *(5)*	$I_{P_2}^G(\omega_{E/F}(\det)\delta^{G_1}[\lambda] \otimes \chi\vert\ \vert_F^{-\lambda})$	E/F *quad. ext.* *include* $E = F^2$ $0 < \lambda < 1/2$	*Table A.2 II.a*
*Th.*4.2 *(5)*	$J_{P_2}^G(\omega_{E/F}(\det)\delta^{G_1}[1/2] \otimes \chi\vert\ \vert_F^{-1/2})$	E/F *quad. ext.* *include* $E = F^2$	*Table A.2 V.b, V.c, VI.b*
*Th.*4.2 *(6)*	$I_{P_2}^G(I_{B_1}^{G_1}(\chi_1 \otimes \chi_1^{-1})[\lambda] \otimes \chi\vert\ \vert_F^{-\lambda})$	$0 < \lambda < 1/2$	*Table A.2 I*
*Th.*4.2 *(5)*	$I_{P_2}^G(\chi_1(\det) \otimes \chi)$	—	*Table A.2 II.b*
*Th.*4.4 *(ii)*	$I_{P_2}^G(\pi[\lambda] \otimes \chi\vert\ \vert_F^{-\lambda})$	$\pi \in \Pi_0(G_1(F))$ $\pi^\vee \simeq \pi$ $0 < \lambda < 1/2$	*Table A.2 X*
*Th.*4.4 *(ii)*	$J_{P_2}^G(\pi[1/2] \otimes \chi\vert\ \vert_F^{-1/2})$	$\pi \in \Pi_0(G_1(F))$ $\pi^\vee \simeq \pi$	*Table A.2 XI.b*

Sketch of the proof. As in the case of $G_1 = GL_2$, we first restrict our consideration to the hermitian classes. Suppose $\pi \in \Pi(G(F))$ and $(\cdot, \cdot)$ is a $G(F)$-invariant inner product on a realization V of π. Then the map

$$A : V \ni v \longmapsto [v' \mapsto (v', v)] \in V^\vee$$

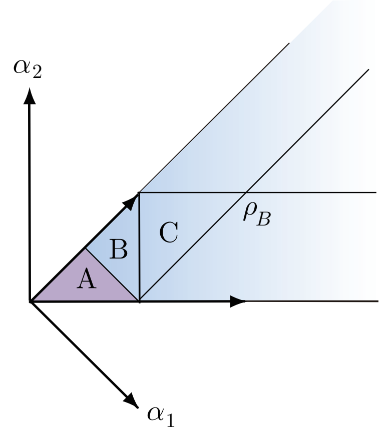
is $G(F)$-invariant $\mathbb{R}$-linear isomorphism satisfying $A(\lambda v) = \bar{\lambda} A(v)$, $(\lambda \in \mathbb{C},\ v \in V)$. An admissible representation (π, V) of $G(F)$ is said to be *hermitian* if there exists a sesquilinear isomorphism $A : \pi \to \pi^\vee$ as above. Obviously, *any unitarizable representation is hermitian.*

We write each $\pi \in \mathrm{Irr}(G(F))$ as a Langlands quotient $\pi \simeq J_P^G(\tau_\lambda)$, $P = MU \in \mathcal{F}(B)$, $\tau \in \Pi_{\mathrm{temp}}(M(F))$, $\lambda \in \mathfrak{a}_P^{*,+}$ in the notation of §1.5. By definition (Th.1.11), this is the unique irreducible submodule of $I_P^G(\tau_\lambda)$, so that $\pi^\vee$ is the unique irreducible quotient of $I_P^G(\tau_{-\lambda}^\vee)$. Now for present G, we have $\bar{P} = w_M^{-1}(P)$ where $w_M \in W$ is given in §3.1. It follows that $\pi^\vee$ is the Langlands quotient of $I^G_{w_M^{-1}(P)}(\tau^\vee_{-\lambda}) \simeq I^G_P(w_M(\tau^\vee_{-\lambda}))$. More explicitly, we have

$$J^G_B(\chi_1 \otimes \chi_2 \otimes \chi)^\vee \simeq J^G_B(\chi_1 \otimes \chi_2 \otimes (\chi_1\chi_2\chi)^{-1}),$$
$$J^G_{P_1}(\chi \otimes \pi)^\vee \simeq J^G_{P_1}(\chi \otimes \chi(\det)^{-1}\pi^\vee), \quad J^G_{P_2}(\pi \otimes \chi)^\vee \simeq J^G_{P_2}(\pi \otimes (\omega_\pi\chi)^{-1}).$$

Since there exists an obvious sesquilinear isomorphism $J^G_P(\tau_\lambda) \longrightarrow J^G_P(\tau^\vee_\lambda)$ (as τ is unitarizable), $J^G_P(\tau_\lambda)$ is hermitian if and only if $w_M(\tau) \simeq \tau$.

Once the irreducible hermitian modules are classified, then we look at the signature of the hermitian forms on such representations. Besides the analysis of the signature illustrated in the GL_2-case, there is a more sophisticated argument using the topology of the unitary dual due to M. Tadić [Tad88]. Here we explain a typical argument in the case of hermitian modules $I^G_B(e^{\boldsymbol{\lambda}})$, $e^{\boldsymbol{\lambda}} = |\ |_F^{\lambda_1} \otimes |\ |_F^{\lambda_2} \otimes |\ |_F^{-\lambda_+}$ with $\lambda_1 > \lambda_2 > 0$, $\lambda_1, \lambda_2 \neq 1$, $\lambda_1 \pm \lambda_2 \neq 1$. The region of $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \mathfrak{a}_T^*$ under consideration is the complement of the reducible lines in the dual positive chamber as illustrated in the picture on the right. From Th.2.7, the three non-compact regions on the right contain no unitarizable representation. Since the representation $I^G_B(\mathbb{1}_{T(F)})$ at the origin is irreducible, the representations which belong to the region A are unitarizable by the same argument as in the proof of Th.2.6(3). At this point, we need the following result of Tadić's theory cited above.



Theorem 5.2 ([Tad86] Th.2.7)**.** *Let $P = MU \in \mathcal{F}_0$ and $\sigma \in \Pi_0(M(F))$. Suppose $I^G_P(\sigma_\lambda)$ is irreducible and unitarizable for any λ in a connected subset $X \subset A_{\widehat{M}}$. Then at λ on the boundary of X, every irreducible subquotient of $I^G_P(\sigma_\lambda)$ are still unitarizable.*

Suppose the region C contains a unitarizable representation. Then the above implies that all the irreducible constituents of $I^G_B(e^{\rho_B})$ must be unitarizable. This contradicts the well-known fact that the irreducible subquotients of $I^G_B(e^{\rho_B})$ other than δ^G_0 and $\mathbb{1}_{G(F)}$ are not unitarizable ([HM79], [BW00, Th.XI.4.5]). Thus C does not contain unitarizable representations.

Finally, note that $I^G_{P_1}(|\ |_F^\lambda \otimes \delta^{G_1}[-\lambda/2])$ is irreducible for $0 < \lambda < 2$ (Th.4.2 (4)). Since the constituent $J^G_{P_1}(|\ |_F^2 \otimes \delta^{G_1}[-1])$ of $I^G_{P_1}(|\ |_F^2 \otimes \delta^{G_1}[-1])$ is not unitarizable, $I^G_{P_1}(|\ |_F^\lambda \otimes \delta^{G_1}[-\lambda/2])$ cannot be unitarizable at any $0 < \lambda < 2$ by Th.5.2. Hence the region B also does not contain any unitarizable class. □

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