On Ikeda's conjecture on the Petersson product of the Ikeda lift

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1 Introduction

For a cuspidal Hecke eigenform f of weight $2k-n$ with respect to $SL_2(\mathbf{Z})$, let $\hat{f}$ be the Ikeda lift of f with respect to $Sp_{2n}(\mathbf{Z})$, and $\tilde{f}$ the modular form in the Kohnen's plus space corresponding to f under the Shimura correspondence. Then Ikeda conjectured that the ratio $\dfrac{\langle \hat{f}, \hat{f} \rangle}{\langle \tilde{f}, \tilde{f} \rangle}$ of the Petersson product of $\hat{f}$ to that of $\tilde{f}$ is expressed by a product of certain L-values of f. In this report, we explain our strategy for proving this conjecture, and report recent progresses, in particular, in case $n=4$. As a related topic, we discuss congruences between Ikeda lifts and non-Ikeda lifts.

Notation.

For a complex number x put $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$.

For a subset S of a commutative ring R, put $S^{\square} = \{a^2; a \in S\}$. For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m,n)-matrices with entries in R. Here we understand $M_{mn}(R)$ the set of the *empty matrix* if $m = 0$ or $n = 0$. In particular put $M_n(R) = M_{nn}(R)$. For an (m, n)-matrix X and an (m, m)-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Let a be an element of R. Then for an element X of $M_{mn}(R)$ we often use the same symbol X to denote the coset $X \bmod aM_{mn}(R)$. Put $GL_m(R) = \{A \in M_m(R); \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix A, and R^* denotes the unit group of R. Let $Sym_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, for an integral domain R of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices of degree n whose (i, j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. In particular, we put $\mathcal{L}_n = \mathcal{H}_n(\mathbf{Z})$. We call an element of $2\mathcal{H}_n(R)$ an even-integral matrix over R. For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. Let R' be a subring of R. Two symmetric matrices A and A' with entries in R are called equivalent over R' with each other and write $A \underset{R'}{\sim} A'$ if there is an element X of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \bot Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

For an integer $D \in \mathbf{Z}$ such that $D \equiv 0$ or $1 \bmod 4$, put $\mathfrak{d}_D$ be the discriminant of $\mathbf{Q}(\sqrt{D})$, and put $\mathfrak{f}_D = \sqrt{\frac{D}{|\mathfrak{d}_D|}}$. Furthermore let χ_D be the character corresponding to $\mathbf{Q}(\sqrt{D})/\mathbf{Q}$. Here we make the convention that $\mathfrak{d}_D = 1$ and $\chi_D = 1$ if $\mathbf{Q}(\sqrt{D}) = \mathbf{Q}$.

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Put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where 1_n denotes the unit matrix of degree n. For a subring K of $\mathbf{R}$ put

$$GSp_{2n}(K)^+ = \{M \in GL_{2n}(K) \ ; \ J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) > 0\},$$

and

$$Sp_{2n}(K) = \{M \in GSp_{2n}(K)^+ \ ; \ J_n[M] = J_n\}.$$

Furthermore, put

$$\mathit{\Gamma}_n = Sp_{2n}(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \ ; \ J_n[M] = J_n\}.$$

Let $\mathbf{H}_n$ be Siegel's upper half-space. Let l be an integer or half integer. For each element $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n(\mathbf{R})^+$ and $Z \in \mathbf{H}_n$ put

$$M(Z) = (AZ + B)(CZ + D)^{-1}$$

and

$$j(M, Z) = \det(CZ + D).$$

Furthermore, for a function F on $\mathbf{H}_n$ we define $F|_l M$ as

$$(F|_l M)(Z) = \det(M)^{l/2} j(M, Z)^{-l} F(M(Z)).$$

For a congruence subgroup $\mathit{\Gamma}$ of $\mathit{\Gamma}_n$, we denote by $M_l(\mathit{\Gamma})$ (resp. $M_l^\infty(\mathit{\Gamma})$ the space of holomorphic (resp. C^∞-) modular forms of weight l with respect to $\mathit{\Gamma}$. For a modular form F of weight l with respect to $\mathit{\Gamma}$ and $\gamma \in \Gamma_n$ we have the following Fourier expansion:

$$F|_l \gamma(Z) = \sum_{A \in \mathcal{L}_{n \geq 0}} a_{\gamma,F}(A) \mathbf{e}(\mathrm{tr}(AZ)),$$

where tr denotes the trace of a matrix. We call $F(Z)$ a cusp form if $a_{\gamma,F}(A) = 0$ unless A is positive-definite. We denote by $S_l(\mathit{\Gamma})$ the suspace of $M_l(\mathit{\Gamma})$ consisting of cusp forms. Let dv denote the invariant volume element on $\mathbf{H}_n$ defined by

$$dv = \det(\mathrm{Im}(Z))^{-n-1} \wedge_{1 \leq i \leq j \leq n} (dx_{ij} \wedge dy_{ij}).$$

Here for $Z \in \mathbf{H}_n$ we write $Z = (x_{ij}) + \sqrt{-1}(y_{ij})$ with real matrices (x_{ij}) and (y_{ij}). For two C^∞-modular forms F and G of weight l with respect to $\mathit{\Gamma}$ we define the Petersson scalar product $\langle F, G \rangle$ by

$$\langle F, G \rangle = [\mathit{\Gamma}_n : \mathit{\Gamma}]^{-1} \int_{\mathit{\Gamma} \backslash \mathbf{H}_n} F(Z)\overline{G(Z)} \det(\mathrm{Im}(Z))^l dv,$$

provided the integral converges.

Let n be a positive integer. For an element $T \in \mathcal{L}_{n>0}$, put $\mathfrak{d}_T = \mathfrak{d}_{(-1)^{n/2}\det(2T)}, \mathfrak{f}_T = \mathfrak{f}_{(-1)^{n/2}\det(2T)}$, and $\chi_T = \chi_{(-1)^{n/2}\det(2T)}$. Now we define the local Siegel series $b_p(T,s)$ by

$$b_p(T,s) = \sum_{R \in Sym_n(\mathbf{Z}[1/p])/Sym_n(\mathbf{Z})} \mathbf{e}(\mathrm{tr}(TR))p^{-ord_p(\mu_p(R))s},$$

where $\mu_p(R) = [R\mathbf{Z}_p^n : \mathbf{Z}_p^n]$. We remark that there exists a unique polynomial $F_p(T,X)$ in X such that $F_p(T,0) = 1$ and

$$b_p(T,s) = F_p(T,p^{-s})\frac{(1-p^{-s})\prod_{i=1}^{n/2}(1-p^{2i-2s})}{1-\chi_T(p)p^{n/2-s}}$$

(cf. Kitaoka [Ki].) Now let k be a positive even integer. Let

$$f(z) = \sum_{m=1}^{\infty} a(m)\mathbf{e}(mz)$$

be a primitive form in $S_{2k-n}(\Gamma_1)$. Furthermore let

$$\tilde{f}(z) = \sum_{e} c(e)\mathbf{e}(ez)$$

be a cuspidal Hecke eigenform in Kohnen's plus subspace $S^+_{k-n/2+1/2}(\Gamma_0(4))$ corresponding to f under the Shimura correspondence (cf. Kohnen, [Ko].) We define a Fourier series $I_n(f)(Z)$ in $Z \in \mathbf{H}_n$ by

$$I_n(f)(Z) = \sum_{T \in \mathcal{L}_{n>0}} a_{I_n(f)}(T)\mathbf{e}(\mathrm{tr}(TZ)),$$

where

$$a_{I_n(f)}(T) = c(|\mathfrak{d}_T|)\prod_p (p^{k-n/2-1/2}\alpha_p)^{ord_p(\mathfrak{f}_T)} \prod_p F_p(T,p^{-(n+1)/2}\alpha_p^{-1}).$$

Then Ikeda [Ik1] showed the following:

$I_n(f)(Z)$ is a Hecke eigenform in $S_k(\Gamma_n)$ whose standard L-function is

$$\zeta(s)\prod_{i=1}^{n} L(s+k-i,f),$$

where $\zeta(s)$ is Riemann's zeta function, and $L(s,f)$ is the L-function of f defined bellow.

We call $I_n(f)$ the Ikeda lift of f. We note that $I_n(f)$ is uniquely determined by $\tilde{f}$. We also note that $I_2(f)$ is the Saito-Kurokawa lift of f.

To formulate Ikeda's conjecture, put

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$

and

$$\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1).$$

We note that

$$\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

Furthermore put

$$\xi(s) = \Gamma_{\mathbf{R}}(s)\zeta(s)$$

and

$$\widetilde{\xi}(s) = \Gamma_{\mathbf{C}}(s)\zeta(s).$$

Let $\alpha_p \in \mathbf{C}$ such that $\alpha_p + \alpha_p^{-1} = p^{-k+n/2+1/2}a(p)$, which we call the Satake p-parameter. For a Dirichlet character χ we define the L-function $L(s,f,\chi)$ of f twisted by χ as

$$L(s,f,\chi) = \prod_p \{(1-\alpha_p p^{-s+k-n/2-1/2}\chi(p))(1-\alpha_p^{-1}p^{-s+k-n/2-1/2}\chi(p))\}^{-1},$$

and put

$$\Lambda(s,\ f,\chi) = \Gamma_{\mathbf{C}}(s)L(s,\ f,\chi).$$

In particular, if χ is the principal character we write $L(s,f,\chi)$ and $\Lambda(s,f,\chi)$ as $L(s,f)$ and $\Lambda(s,f)$, respectively. Furthermore, we define the adjoint L-function $L(s,\ f,\ \mathrm{Ad})$ as

$$L(s,\ f,\ \mathrm{Ad}) = \prod_{p\,:\,\mathrm{prime}} \{(1-p^{-s})(1-\alpha_p^2 p^{-s})(1-\alpha_p^{-2}\,p^{-s})\}^{-1},$$

and put

$$\widetilde{\Lambda}(s,\ f,\ \mathrm{Ad}) = \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s+2k-n-1)L(s,\ f,\ \mathrm{Ad}).$$

Now we have the following diagram of liftings:

$$\begin{array}{ccccc} S^+_{k-(n-1)/2}(\mathit{\Gamma}_0(4)) & \leftrightarrow & S_{2k-n}(\mathit{\Gamma}_1) & \rightarrow & S_k(\mathit{\Gamma}_n) \\ \tilde{f} & \leftrightarrow & f & \mapsto & I_n(f) \end{array}$$

Then Ikeda[Ik2] proposed the following conjecture:

Conjecture A. There exists an integer $\alpha(n,\,k)$ such that

$$\frac{\langle I_n(f),\, I_n(f)\rangle}{\langle \tilde{f},\, \tilde{f}\rangle} = 2^{\alpha(n,\,k)}\Lambda(k,\; f)\widetilde{\xi}(n)\prod_{i=1}^{n/2-1}\tilde{\Lambda}(2i+1,\; f,\; \mathrm{Ad})\,\widetilde{\xi}(2i)$$

Remark 1. When $n=2$, Conjecture A is true. Namely, Kohnen and Skoruppa showed that

$$\frac{\langle I_2(f),\, I_2(f)\rangle}{\langle \tilde{f},\, \tilde{f}\rangle} = 2^{k-1}\Lambda(k,\; f)\widetilde{\xi}(2).$$

Remark 2. By the result in Kohnen-Zagier[K-Z], for any fundamental discriminant D such that $(-1)^{n/2}D>0$ we have

$$\frac{c(|D|)^2}{\langle \tilde{f}, \tilde{f}\rangle} = \frac{2^{k-n/2-1}|D|^{k-n/2-1/2}\Lambda(k-n/2, f, \chi_D)}{\langle f, f\rangle}.$$

Thus, assuming Conjecture A, for any fundamental discriminant D such that $(-1)^{n/2}D>0$ and $L(k-n/2, f, \chi_D)\neq 0$ we have

$$\frac{\langle I_n(f),\, I_n(f)\rangle}{\langle f,\; f\rangle^{n/2}} = \frac{a_{n,k}c(|D|)^2\Lambda(k,\; f)}{|D|^{k-n/2-1/2}\Lambda(k-n/2, f, \chi_D)}\widetilde{\xi}(n)\prod_{i=1}^{n/2-1}\frac{\widetilde{\Lambda}(2i+1,\; f,\; \mathrm{Ad})}{\langle f,\; f\rangle}\widetilde{\xi}(2i)$$

with some algebraic number $a_{n,k}$ depending only on n,k. It is well-known that $\dfrac{\Lambda(k,\; f)}{\Lambda(k-n/2, f, \chi_D)}$ and $\dfrac{\widetilde{\Lambda}(2i+1,\; f,\; \mathrm{Ad})}{\langle f,\; f\rangle}$ for $i=1,...,n/2-1$ are algebraic number and belong to the Hecke field $\mathbf{Q}(f)$. (cf. Boecherer[Bo2], Shimura[Sh1], [Sh4].) Thus if all the Fourier coefficients of $\tilde{f}$ are algebraic, then $\dfrac{\langle I_n(f),\, I_n(f)\rangle}{\langle f,\; f\rangle^{n/2}}$ is algebraic (cf. Furusawa[F], Choie-Kohnen[C-K].)

3 Rankin-Selberg Dirichlet series associated with the Fourier-Jacobi expansion of the Ikeda lift

To prove Conjecture A, first we consider a certain Rankin-Selberg Dirichlet series associated with the Fourier-Jacobi expansion of the Ikeda lift. Let

$F \in S_k(\varGamma_n)$. Then we have the following Fourier expansion:

$$F(Z) = \sum_{B \in \mathcal{L}_{n>0}} A(B)\mathbf{e}(\mathrm{tr}(BZ)) \quad (Z \in \mathbf{H}_n).$$

Write $Z = \begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}$ with $\tau \in \mathbf{H}_{n-1}$, $z \in \mathbf{C}^{n-1}$ and $\tau' \in \mathbf{H}_1$, and we have the Fourier-Jacobi expansion of F of type $(1, n-1)$ as follows:

$$F\left(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}\right) = \sum_{N=1}^{\infty} \phi_N(\tau, z)\mathbf{e}(N\tau'),$$

where $\phi_N(\tau, z)$ is the N-th Fourier-Jacobi coefficient of F and defined as follows:

$$\phi_N(\tau, z) = \sum_{\substack{T \in \mathcal{L}_{n-1}, r \in \mathbf{Z}^{n-1}, \\ 4NT - {}^t rr > 0}} A\left(\begin{pmatrix} N & r/2 \\ {}^t r/2 & T \end{pmatrix}\right) \mathbf{e}(\mathrm{tr}(T\tau) + r^t z).$$

Let $J_{k,N}^{\mathrm{cusp}}(\varGamma_{n-1}^J)$ denote the space of Jacobi cusp forms of weight k and index N with respect to $\varGamma_{n-1}$. Then $\phi_N \in J_{k,N}^{\mathrm{cusp}}(\varGamma_{n-1}^J)$ for each $N \in \mathbf{Z}_{>0}$.

Now we define a Dirichlet series $D_1(s, F)$ as

$$D_1(s; F) = \zeta(2s - 2k + 2n) \sum_{N=1}^{\infty} \langle \phi_N, \phi_N \rangle N^{-s},$$

where $\langle \phi_N, \phi_N \rangle$ is the Petersson product defined on the space $J_{k,N}^{\mathrm{cusp}}(\varGamma_{n-1}^J)$. Then, as for the analytic properties of $D_1(s; F)$ we refer to the following result due to Yamazaki [Y2]:

Let $\Gamma_{n,k}(s) = \pi^{k-n}(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+n)$, *Then the function*

$$\mathcal{D}_1(s; F) := \Gamma_{n,k}(s)\, D_1(s; F)$$

has a meromorphic continuation to the whole s-plane, and has simple poles at $s = k$ *and* $s = k - n$ *with the residue* $\langle F, F \rangle$. *Furthermore, it satisfies the following functional equation:*

$$\mathcal{D}_1(s; F) = \mathcal{D}_1(2k - n - s; F).$$

Now we give an explicit formula for $D_1(s, I_n(f))$:

Thmeorem. 3.1. ([Kat-Kaw1]) *Let n and k be positive even integers s.t. $k > n+1$. Let f be a primitive form in $S_{2k-n}(\Gamma_1)$, and $\phi_1 = \phi_{I_n(f),1}$ the first Fourier-Jacobi coefficient of $I_n(f)$. Then we have*

$$D_1(s; I_n(f))$$

$$= \langle \phi_1,\, \phi_1 \rangle\, \zeta(s-k+1)\zeta(s-k+n)L(s,\, f).$$

We give an outline of the proof of Theorem 3.1.
Let

$$I_n(f)\left(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}\right) = \sum_{N=1}^{\infty} \phi_N(\tau,\, z)\mathbf{e}(N\tau').$$

First we use the following fact due to Hayashida:

Fact 1. (Hayashida[Ha]) *For each N and m, there is a homorophism*

$$D_f(N) : J_{k,\,m}^{\mathrm{cusp}}(\Gamma_{n-1}^J) \longrightarrow J_{k,\,mN}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$$

such that $D_f(N)(\phi_m) = \phi_{mN}$.

We note that $D_f(N)$ coincides with the usual shift operator V_N in Eichler-Zagier [E-Z] and with $D_{n-1}(N)$ in Yamazaki[Y1] in case $n = 2$. However it does not so in genaral, and depends on f. Next we use the fact concerning the adjoint operator of $D_f(N)$. To explain it more precisely, for a positive integer N

$$\Psi_p(M;\, \alpha_p) = \frac{\alpha_p^{\delta+1} - \alpha_p^{-(\delta+1)}}{\alpha_p - \alpha_p^{-1}} + p^{-(n-1)/2} \cdot \frac{\alpha_p^{\delta} - \alpha_p^{-\delta}}{\alpha_p - \alpha_p^{-1}}$$

where $\delta = \mathrm{ord}_p(M)$, and α_p is the Satake p-parameter. Then we have the following:

Fact 2. Let $D_f^*(N) : J_{k,\,mN}^{\mathrm{cusp}}(\Gamma_{n-1}^J) \longrightarrow J_{k,\,N}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$ be the adjoint operator of $D_f(N)$, that is

$$\langle D_f(N)(\phi),\, \psi \rangle = \langle \phi,\, D_f^*(N)(\psi) \rangle$$

for any $\phi \in J_{k,m}^{\text{cusp}}(\Gamma_{n-1}^J)$ and $\psi \in J_{k,mN}^{\text{cusp}}(\Gamma_{n-1}^J)$. Then

$$D_f^*(N)D_f(N)(\phi_1) = \sum_{d|N} d^{k-1}(N/d)^{k-(n+1)/2} \prod_{p|N/d} \Psi_p(N/d;\alpha_p)\phi_1.$$

Now by using the above two facts we have

$$\sum_{N=1}^{\infty} \langle \phi_N, \phi_N \rangle N^{-s} = \sum_{N=1}^{\infty} \langle D_f(N)(\phi_1), D_f(N)(\phi_1) \rangle N^{-s}$$

$$= \sum_{N=1}^{\infty} \langle \phi_1, D_f^*(N)D_f(N)(\phi_1) \rangle N^{-s}$$

$$= \sum_{N=1}^{\infty} \langle \phi_1, \phi_1 \rangle N^{-s} \sum_{d|N} d^{k-1}(N/d)^{k-(n+1)/2} \prod_{p|N/p} \Psi_p(N/d;\alpha_p)$$

$$= \langle \phi_1,\ \phi_1 \rangle \frac{\zeta(s-k+1)\zeta(s-k+n)L(s,\ f)}{\zeta(2s-2k+2n)}.$$

Thus the assertion holds.

By taking the residues of the both sides of Theorem 3.1, we have

Corollary. Under the same assumption as above, we have

$$\frac{\langle I_n(f),\ I_n(f) \rangle}{\langle \phi_1,\ \phi_1 \rangle} = 2^{-k+n-2}\Lambda(k,\ f)\widetilde{\xi}(n). \tag{1}$$

4 Rankin-Selberg convolution product of the Ibukiyama correspondence

To prove Conjecture B, we consider the Rankin-Selberg convolution product of the Ibukiyama correspondence. Let

$$\Gamma_0^{(n-1)}(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{n-1}\ ;\ C \equiv O \bmod N\}.$$

Let l be a positive half integer. Let $F(Z) \in S_l(\Gamma_0^{(n-1)}(4))$. Then $F(Z)$ has the following Fourier expansion:

$$F(Z) = \sum_{A \in \mathcal{L}_{n-1>0}} a_F(A)\mathbf{e}(\mathrm{tr}(AZ))$$

We define the Rankin-Selberg convolution product $R(s,F)$ of F as

$$R(s,F) = \sum_{A \in \mathcal{L}_{n-1>0}/SL_{n-1}(\mathbf{Z})} \frac{|a_F(A)|^2}{e(A)(\det A)^s},$$

where $e(A) = \#\{X \in SL_{n-1}(\mathbf{Z});\ {}^tXAX = A\}$. As for the analytic properties of $R(s,F)$, we have the following:

Proposition 4.1. *Let $F \in S_l(\varGamma_0^{(n-1)}(4))$. Put*

$$\gamma_{n-1}(s) = 2^{1-2s(n-1)}\pi^{-s(n-1)+(n-1)(n-2)/4}\prod_{j=1}^{n-1}\Gamma(s-(j-1)/2),$$

and

$$c_n = [\Gamma_{n-1} : \Gamma_0^{(n-1)}(4)]4^{-n(n-1)/2}2^{-1}\frac{\prod_{i=1}^{(n-2)/2}(1-2^{-2i-1})}{(1-2^{-n})\prod_{i=1}^{(n-2)/2}(1-2^{2i-2n})}.$$

Then $R(s,F)$ has a meromorphic continuation to the whole s-plane, and has a simple pole at $s=l$ with the residue

$$c_n\gamma_{n-1}(l)^{-1}\frac{\prod_{i=1}^{(n-2)/2}\xi(2i+1)}{\xi(n)\prod_{i=1}^{(n-2)/2}\xi(2n-2i)}\langle F, F\rangle.$$

Proof. The assertion can be proved in the same manner as in Kalinin[Kal] combined with the result of Shimura. But, for the readers' convenience we here give an outline of the proof. For a positive integer, we define the non-holomorphic Siegel Eisenstein series $E_N(Z,s)$ by

$$E_N(Z,s) = \det \mathrm{Im}(Z)^s \sum_{M \in \varGamma_0^{(n-1)}(N)_\infty \backslash \varGamma_0^{(n-1)}(4)} |j(M,Z)|^{-2s},$$

where $\varGamma_0^{(n-1)}(N)_\infty = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \varGamma_0^{(n-1)}(N)\}$. Then by the usual Rankin-Selberg method, we have

$$R(s,F) = \gamma_{n-1}(s)^{-1}[\varGamma_n : \varGamma_0^{(n-1)}(4)]\langle FE_4(*, s+n/2-l), F\rangle.$$

On the other hand, by a careful analysis of Shimura[Sh4], the function

$$\xi_N(2s+n-2l)\prod_{i=1}^{(n-2)/2}\xi_N(4s+2n-4l-2i)E_N(Z,s+n-l)$$

has a meromorphic continuation to the whole s-plane, and has a simple pole at $s=l$ with the residue

$$4^{-n(n-1)/2}\prod_{p|N}(1-p^{-1})\prod_{i=1}^{(n-2)/2}\xi_N(2j+1),$$

where $\xi_N(s)=\prod_{p|N}(1-p^{-s})\xi(s)$. Thus the assertion holds.

Let k be a positive integer. We then define the generalized Kohnen's plus subspace of weigth $k-1/2$ with respective to $\Gamma_0^{(n-1)}(4)$ as

$$S_{k-1/2}^{+}(\Gamma_0^{(n-1)}(4))=\{F(Z)=\sum_{A\in\mathcal{L}_{n-1>0}}c(A)\mathbf{e}(\mathrm{tr}(AZ))\in S_{k-1/2}(\Gamma_0^{(n-1)}(4));$$

$$c(A)=0 \text{ unless } A\equiv(-1)^{k+1}\,{}^t rr \bmod 4\mathcal{L}_{n-1}\text{for some } r\in\mathbf{Z}^{n-1}\}$$

Then there exsits a correspondence between the space of Jacobi-forms of index 1 and generalized Kohnen's plus space due to Ibukiyama. To explain this, let $\phi(Z,z)\in J_{k,1}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$. Then $\phi(\tau,z)$ can be expressed as follows:

$$\phi(\tau,z)=\sum_{r\in\mathbf{Z}^{n-1}/2\mathbf{Z}^{n-1}}h_r(\tau)\theta_r(\tau,z),$$

where $h_r(\tau)$ is a holomorphic function on $\mathbf{H}_{n-1}$, and

$$\theta_r(\tau,z)=\sum_{\lambda\in M_{1,n-1}(\mathbf{Z})}\mathbf{e}(\mathrm{tr}(\tau[{}^t(\lambda+2^{-1}r)]+2{}^t(\lambda+2^{-1}r)z).$$

We note that having the following Fourier-Jacobi expansion,

$$\phi(\tau,\,z)=\sum_{\substack{T\in\mathcal{L}_{n-1},\,r\in\mathbf{Z}^{n-1},\\ 4T-\,{}^t rr>0}}c(T,r)\mathbf{e}(\mathrm{tr}(T\tau)+r\,{}^t z),$$

we have

$$h_r(\tau)=\sum_{A\in\frac{1}{4}Sym_{n-1}(\mathbf{Z})_{>0}}c(A+{}^t rr/4,r)\mathbf{e}(\mathrm{tr}(A\tau)).$$

We then put

$$H_{n-1}(\phi) = \sum_{r \in \mathbf{Z}^{n-1}/2\mathbf{Z}^{n-1}} h_r(4\tau).$$

Then Ibukiyama[Ib] showed the following:

Let k be even positive integer. Then H_{n-1} gives a $\mathbf{C}$-linear Hecke equivariant isomorphism

$$H_{n-1} : J_{k,1}^{\mathrm{cusp}}(\mathit{\Gamma}_{n-1}^J) \cong S_{k-1/2}^{+}(\mathit{\Gamma}_0^{(n-1)}(4)).$$

We call H_{n-1} the Ibukiyama correspondence. We note that we have

$$H_{n-1}(\phi) = \sum_{A \in Sym_{n-1}(\mathbf{Z})_{>0}} \sum_{r \in \mathbf{Z}^{n-1}/2\mathbf{Z}^{n-1}} c((A + {}^t rr)/4, r)\mathbf{e}(\mathrm{tr}(A\tau)).$$

Put $\mathbf{h}(\tau) = (h_r(\tau))_{r \in \mathbf{Z}^{n-1}/2\mathbf{Z}^{n-1}}$. Then $\mathbf{h}$ is a vector valued modular form with respect to Γ_{n-1}. Define a Dirichlet series $R(s, \mathbf{h})$ as

$$R(s, \mathbf{h}) = \sum_{A \in \frac{1}{4} Sym_{n-1}(\mathbf{Z})_{>0}/SL_{n-1}(\mathbf{Z})} \sum_{r \in \mathbf{Z}^{n-1}/2\mathbf{Z}^{n-1}} \frac{|c(A + \ {}^t rr/4, r)|^2}{e(A) \det A^s}.$$

As for the relation between the Petersson products, we have the following:

Proposition 4.2. We have

$$\langle \phi, \phi \rangle = c_n 2^{(2k-2)(n-1)} \langle H_{n-1}(\phi), H_{n-1}(\phi) \rangle,$$

for any $\phi \in J_{k,1}^{\mathrm{cusp}}(\mathit{\Gamma}_{n-1}^J)$, where c_n is the constant in Proposition 4.1.

Proof. It is well-known that we have

$$\langle \phi, \phi \rangle = 2^{1-n} \sum_{r \in \mathbf{Z}^{n-1}/2\mathbf{Z}^{n-1}} \int_{\Gamma_{n-1} \backslash \mathbf{H}_{n-1}} h_r(\tau)\overline{h_r(\tau)} \mathrm{Im}(\tau)^{k-1/2} dv,$$

where $h_r(\tau)$ is the function stated above. Since $\mathbf{h}$ is a vector valued modular form with respect to Γ_{n-1}, we can again apply the Rankin-Selberg method and we have the following:

$$R(s, \mathbf{h}) = \gamma_{n-1}(s)^{-1} \sum_{r \in \mathbf{Z}^{n-1}/2\mathbf{Z}^{n-1}} \int_{\Gamma_{n-1} \backslash \mathbf{H}_{n-1}} h_r(\tau)\overline{h_r(\tau)} \mathrm{Im}(\tau)^{k-1/2} E_1(\tau, z) dv,$$

where $\gamma_{n-1}(s)$ is the function in Proposition 4.1. Thus by taking the residue, we have

$$\mathrm{Res}_{s=k-1/2}R(s,\mathbf{h}) = 2^{n-1}\gamma_{n-1}(k-1/2)^{-1}\frac{\prod_{i=1}^{(n-2)/2}\xi(2i+1)}{\xi(n)\prod_{i=1}^{(n-2)/2}\xi(2n-2i)}\langle\phi,\phi\rangle.$$

We note that we have

$$R(s,\mathbf{h}) = 2^{2s(n-1)}R(s,H_{n-1}(\phi)).$$

Thus by Theorem 4.1, the assertion holds.

Now we have the following diagram of liftings:

$$\begin{array}{ccc} S^+_{k-(n-1)/2}(\mathit{\Gamma}_0^{(1)}(4)) \ni \tilde{f} & \longrightarrow & f \in S_{2k-n}(\mathit{\Gamma}_1) \\ & & \downarrow \\ & & I_n(f) \in S_k(\mathit{\Gamma}_n) \\ & & \downarrow \\ S^+_{k-1/2}(\mathit{\Gamma}_0^{(n-1)}(4)) \ni H_{n-1}(\phi_1) & \longleftarrow & \phi_1 \in J^{\mathrm{cusp}}_{k,1}(\mathit{\Gamma}^J_{n-1}) \end{array}$$

Then by using Corollary to Theorem 3.1 and Proposition 4.2, we see that the Conjecture A is equivalent to the following:

Conjecture B Assume the same situation as above. Then there exists an integer $\beta(n,\,k)$ depending only on n and k such that

$$\frac{\langle H_{n-1}(\phi_1),\, H_{n-1}(\phi_1)\rangle}{\langle \tilde{f},\, \tilde{f}\rangle} = c_n^{-1}2^{\beta(n,\,k)}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)\,\widetilde{\Lambda}(2i+1,\,f,\,\mathrm{Ad})$$

Remark. *Assuming Conjecture B,* $\dfrac{\langle H_{n-1}(\phi_1),\, H_{n-1}(\phi_1)\rangle}{\langle \tilde{f},\, \tilde{f}\rangle\langle f, f\rangle^{n/2-1}}$ *is an algebraic number and can be computed exactly.*

Now we propose one conjecture concerning an explicit form of $R(s, H_{n-1}(\phi))$ for the first Fourier-Jacobi coefficient ϕ of the Ikeda lift:

Conjecture C. *For a primitive form* $f \in S_{2k-n}(\Gamma_1)$, *let* $\widetilde{f} \in S^+_{k-n/2+1/2}(\Gamma_0(4))$ *and* $\phi_1 = \phi_{I_n(f),1} \in J^{\mathrm{cusp}}_{k,1}(\Gamma^J_{n-1})$ *be as above. Put* $\lambda_n = \frac{1}{2}\prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)$. *Then, we have*

$$R(s, H_{n-1}(\phi_1)) = \lambda_n 2^{a_{n,k}+b_{n,k}s}\zeta(2s+n-2k+1)^{-1}\prod_{i=1}^{\frac{n-2}{2}}\zeta(4s+2n-4k+2-2j)^{-1}$$

$$\times\{R(s-n/2+1, \tilde{f})\zeta(2s-2k+3)\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+2, f, \mathrm{Ad})\zeta(2s-2k+2i+2)$$

$$+R(s, \tilde{f})\zeta(2s-2k+n+1)\prod_{i=1}^{\frac{n-2}{2}}L(2s-2k+2i+1, f, \mathrm{Ad})\zeta(2s-2k+2i+1)\},$$

where $a_{n,k}$ *and* $b_{n,k}$ *be integers depending only on* n *and* k.

Remark. The right-hand side of the above form multiplied by

$$\gamma_{n-1}(s)\xi(2s+n-2k+1)\prod_{i=1}^{(n-2)/2}\xi(4s+2n-4k+2-2i)$$

is invariant under $s \longrightarrow 2k-1-n/2-s$.

Theorem 4.3. *If Conjecture C is true, so is Conjecture B.*

Proof. By taking the residue of the both-sides at $s = k-1/2$ in Conjecture C, we get

$$\mathrm{Res}_{s=k-1/2}R(s, H_{n-1}(\phi_1)) = \prod_{i=1}^{n/2-1}\widetilde{\xi}(2i)\zeta(n)^{-1}\prod_{i=1}^{\frac{n-2}{2}}\zeta(2n-2j)^{-1}$$

$$\times\{\mathrm{Res}_{s=k-n/2+1/2}R(s, \tilde{f})\zeta(2)\prod_{i=1}^{\frac{n-2}{2}}L(2i+1, f, \mathrm{Ad})\zeta(2i+1)\}.$$

On the other hand, by Proposition 4.1, we have

$$c_n\gamma_{n-1}(k-1/2)^{-1}\frac{\prod_{i=1}^{(n-2)/2}\xi(2i+1)}{\xi(n)\prod_{i=1}^{(n-2)/2}\xi(2n-2i)}\langle H_{n-1}(\phi_1), H_{n-1}(\phi_1)\rangle$$

$$= \lambda_n \zeta(n)^{-1} \prod_{i=1}^{\frac{n-2}{2}} \zeta(2n-2j)^{-1}$$

$$\times c_2 \gamma_1(k-n/2+1/2)^{-1} \xi(2)^{-1} \langle \tilde{f}, \tilde{f} \rangle \zeta(2) \prod_{i=1}^{\frac{n-2}{2}} L(2i+1, f, \mathrm{Ad}) \zeta(2i+1)\}.$$

We note that

$$\zeta(s) = \Gamma_{\mathbf{R}}(s)^{-1} \xi(s),$$

and

$$L(s, f, \mathrm{Ad}) = \Gamma_{\mathbf{C}}(s)^{-1} \Gamma_{\mathbf{C}}(s+2k-n-1)^{-1} \widetilde{\Lambda}(s,\ f,\ \mathrm{Ad}).$$

We also note that

$$\gamma_{n-1}(k-1/2)/\gamma_1(k-n/2+1/2) = 2^{-2k(n-2)} \prod_{i=1}^{(n-2)/2} \Gamma_{\mathbf{C}}(2k-n+2i).$$

Thus we have

$$\frac{c_n \langle H_{n-1}(\phi_1), H_{n-1}(\phi_1) \rangle}{c_2 \langle \tilde{f}, \tilde{f} \rangle \prod_{i=1}^{\frac{n-2}{2}} \{L(2i+1, f, \mathrm{Ad}) \widetilde{\xi}(2i)\}} = \frac{2^{-2k(n-2)} \Gamma_{\mathbf{R}}(n) \prod_{i=1}^{(n-2)/2} \Gamma_{\mathbf{R}}(2n-2i)}{\Gamma_{\mathbf{R}}(2) \prod_{i=1}^{(n-2)/2} \Gamma_{\mathbf{C}}(2i+1) \Gamma_{\mathbf{R}}(2i+1)}$$

$$= \frac{2^{-2k(n-2)} \prod_{i=2}^{n-1} \Gamma_{\mathbf{R}}(2i)}{\prod_{i=1}^{(n-2)/2} \Gamma_{\mathbf{C}}(2i+1) \Gamma_{\mathbf{R}}(2i+1) \Gamma_{\mathbf{R}}(2i)} = \frac{2^{-2k(n-2)} \prod_{i=2}^{(n-1)} 2^{i-1} \Gamma_{\mathbf{C}}(i)}{\prod_{i=1}^{(n-2)/2} \Gamma_{\mathbf{C}}(2i+1) \Gamma_{\mathbf{C}}(2i)}$$

$$= 2^{-2k(n-2)+(n-2)(n-1)/2}.$$

This proves the assertion.

Theorem 4.4. ([Kat-Kaw2]) *Conjecture C is true for* $n=4$, *and therefore, so is Conjecture A.*

The proof of Theorem 4.4 is similar to those in the main results in [Ib-Kat1],[Ib-Kat2], and can be reduced to a computation of certain formal power series attached to polynomilas $F(B, X)$ in Section 2. However the situation is more complex than those.

5 Congruences between Ikeda lifts and non-Ikeda lifts

In this section, we consider the congruence between Ikeda lifts and non-Ikeda lifts. Let $\mathbf{L}_n = \mathbf{L}_{\mathbf{Q}}(GSp_{2n}(\mathbf{Q})^+, \mathit{\Gamma}_n)$ denote the Hecke ring over $\mathbf{Q}$ associated with the Hecke pair $(GSp_{2n}(\mathbf{Q})^+, \mathit{\Gamma}_n)$. For each integer m define an element $T(m)$ of $\mathbf{L}_n$ by

$$T(m) = \sum_{d_1,\ldots,d_n,e_1,\ldots,e_n} \mathit{\Gamma}_n(d_1 \perp \ldots \perp d_n \perp e_1 \perp \ldots \perp e_n)\mathit{\Gamma}_n,$$

where $d_1, ..., d_n, e_1, ..., e_n$ run over all positive integer satisfying

$$d_i | d_{i+1},\ e_{i+1} | e_i\ (i = 1, ..., n-1), d_n | e_n, d_i e_i = m\ (i = 1, ..., n).$$

Furthermore, for $i = 0, 1, ..., n$ and a prime number p not dividing N, put

$$T_i(p^2) = \mathit{\Gamma}_n(1_{n-i} \perp p1_i \perp p^2 1_{n-i} \perp p1_i)\mathit{\Gamma}_n.$$

As is well known, $\mathbf{L}_n$ is generated over $\mathbf{Q}$ by all $T(p)$ and $T_i(p^2)$ $(i = 1, ..., n)$. We denote by $\mathbf{L}'_n$ the subalgebra of $\mathbf{L}_n$ generated by over $\mathbf{Z}$ by all $T(p)$ and $T_i(p^2)$ $(i = 1, ..., n)$. Let $T = \mathit{\Gamma}_n M \mathit{\Gamma}_n$ be an element of $\mathbf{L}_n \otimes \mathbf{C}$. Write T as $T = \cup_\gamma \mathit{\Gamma}_n \gamma$ and for $f \in M_k(\mathit{\Gamma}_n)$ define the Hecke operator $|_k T$ associated to T as

$$f|_k T = \det(M)^{k/2-(n+1)/2} \sum_\gamma f|_k \gamma.$$

We call this action the Hecke operator as usual (cf. [A].) If f is an eigenfunction of a Hecke operator $T \in \mathbf{L}_n \otimes \mathbf{C}$, we denote by $\lambda_f(T)$ its eigenvalue. We call $f \in M_k(\mathit{\Gamma}_n)$ a Hecke eigenform if it is a common eigenfunction of all Hecke operators. Furthermore, we denote by $\mathbf{Q}(f)$ the field generated over $\mathbf{Q}$ by eigenvalues of all $T \in \mathbf{L}_n$ as in Section 2. As is well known, $\mathbf{Q}(f)$ is a totally real algebraic number field of finite degree. Now, first we consider the integrality of the eigenvalues of Hecke operators. For an algebraic number field K, let $\mathfrak{O}_K$ denote the ring of integers in K.

Proposition 5.1 *Let $k \geq n+1$. Let $f \in S_k(\mathit{\Gamma}_n)$ be a common eigenfunction of all Hecke operators in $\mathbf{L}'_n$. Then $\lambda_f(T)$ belongs to $\mathfrak{O}_{\mathbf{Q}(f)}$ for any $T \in \mathbf{L}'_n$.*

The above theorem can be proved by using [H-W] and [Sh2]. For the details, see Proposition 4.1 of [Kat4].

Now we review a result concerning the congruence between the Hecke eigenvalues of modular forms of the same weight following [Kat4] and [Kat5]. Let K be an algebraic number field, and $\mathfrak{O} = \mathfrak{O}_K$ the ring of integers in K. For a prime ideal $\mathfrak{P}$ of $\mathfrak{O}$, we denote by $\mathfrak{O}_{(\mathfrak{P})}$ the localization of $\mathfrak{O}$ at $\mathfrak{P}$ in K. Let $\mathfrak{A}$ be a fractional ideal in K. If $\mathfrak{A} = \mathfrak{P}^e\mathfrak{B}$ with $\mathfrak{B}\mathfrak{O}_{(\mathfrak{P})} = \mathfrak{O}_{(\mathfrak{P})}$ we write $\mathrm{ord}_{\mathfrak{P}} = e$. We simply write $\mathrm{ord}_{\mathfrak{P}}(c) = \mathrm{ord}_{\mathfrak{P}}((c))$ for $c \in K$. Let f be a Hecke eigenform in $S_k(\Gamma_n)$ and M be a subspace of $S_k(\Gamma_n)$ stable under Hecke operators $T \in \mathbf{L}_n$. Assume that M is contained in $(\mathbf{C}f)^\perp$, where $(\mathbf{C}f)^\perp$ is the orthogonal complement of $\mathbf{C}f$ in $S_k(\Gamma_n)$ with respect to the Petersson product. Let K be an algebraic number field containing $\mathbf{Q}(f)$. A prime ideal $\mathfrak{P}$ of $\mathfrak{O}_K$ is called a congruence prime of f with respect to M if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \bmod \tilde{\mathfrak{P}}$$

for any $T \in \mathbf{L}'_n$, where $\tilde{\mathfrak{P}}$ is the prime ideal of $\mathfrak{O}_{K\mathbf{Q}(g)}$ lying above $\mathfrak{P}$. If $M = (\mathbf{C}f)^\perp$, we simply call $\mathfrak{P}$ a congruence prime of f.

Now returning to the case of the Ikeda lift, we propose one conjecture concerning the congruence between Ikeda lifts and non-Ikeda lifts. Let f be a primitive form in $S_{2k-n}(\Gamma_1)$. Let $\{f_1, ..., f_d\}$ be a basis of $S_{2k-n}(\Gamma_1)$ consisting of primitive forms. Let K be an algebraic number field containing $\mathbf{Q}(f_1)\cdots\mathbf{Q}(f_d)$, and $A = \mathfrak{O}_K$. To formulate our conjecture exactly, we introduce the Eichler-Shimura periods as follows (cf. Hida [Hi3].) Let $\mathfrak{P}$ be a prime ideal in K. Let $A_{\mathfrak{P}}$ be a valuation ring in K corresponding to $\mathfrak{P}$. Assume that the residual characteristic of $A_{\mathfrak{P}}$ is greater than or equal to 5. Let $L(2k-n-2, A_{\mathfrak{P}})$ be the module of homogeneous polynomials of degree $2k-n-2$ in the variables X, Y with coefficients in $A_{\mathfrak{P}}$. We define the action of $M_2(\mathbf{Z}) \cap GL_2(\mathbf{Q})$ on $L(2k-n-2, A_{\mathfrak{P}})$ via

$$\gamma \cdot P(X,Y) = P({}^t(X,Y)(\gamma)^\iota),$$

where $\gamma^\iota = (\det\gamma)\gamma^{-1}$. Let $H^1_P(\Gamma_1, L(2k-n-2, A_{\mathfrak{P}}))$ be the parabolic cohomology group of Γ_1 with values in $L(2k-n-2, A_{\mathfrak{P}})$. Fix a point $z_0 \in \mathbf{H}_1$. Let $g \in S_{2k-n}(\Gamma_1)$ or $g \in \overline{S_{2k-n}(\Gamma_1)}$. We then define the differential $\omega(g)$ as

$$\omega(g)(z) = \begin{cases} 2\pi i g(z)(X - zY)^n dz & \text{if } g \in S_{2k-n}(\Gamma_1) \\ 2\pi i g(z)(X - \bar{z}Y)^n dz & \text{if } g \in \overline{S_{2k-n}(\Gamma_1)}, \end{cases}$$

and define the cohomology class $\delta(g)$ of the 1-cocycle of Γ_1. as

$$\gamma \in \Gamma_1 \longrightarrow \int_{z_0}^{\gamma(z_0)} \omega(g).$$

The mapping $g \longrightarrow \delta(g)$ induces the isomorphism

$$\delta : S_{2k-n}(\Gamma_1) \oplus \overline{S_{2k-n}(\Gamma_1)} \longrightarrow H_P^1(\Gamma_1, L(2k-n-2, \mathbf{C})),$$

which is called Eichler-Shimura isomorphism. We can define the action of Hecke algebra $\mathbf{L}_1'$ on $H_P^1(\Gamma_1, L(2k-n-2, A_{\mathfrak{P}}))$ in a natural manner. Furthermore, we can define the action F_∞ on $H_P^1(\Gamma_1, L(2k-n-2, A_{\mathfrak{P}}))$ as

$$F_\infty(\delta(g)(z)) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \delta(g)(-\bar{z}),$$

and this action commutes with the Hecke action. For a primitive form f and $j = \pm 1$, we define the subspace $H_P^1(\Gamma_1, L(2k-n-2, A_{\mathfrak{P}}))[f,j]$ of $H_P^1(\Gamma_1, L(2k-n-2, A_{\mathfrak{P}}))$ as

$$H_P^1(\Gamma_1, L(2k-n-2, A_{\mathfrak{P}}))[f,j]$$

$$= \{x \in H_P^1(\Gamma_1, L(2k-n-2, A_{\mathfrak{P}}))\ ;\ x|T = \lambda_f(T)x \text{ for } T \in \mathbf{L}_1, \text{ and } F_\infty(x) = jx\}.$$

Since $A_{\mathfrak{P}}$ is a principal ideal domain, $H_P^1(\Gamma_1, L(2k-n-2, A_{\mathfrak{P}}))[f,j]$ is a free module of rank one over $A_{\mathfrak{P}}$. Take a basis $\eta(f,j,A_{\mathfrak{P}})$ of $H_P^1(\Gamma_1,(2k-n-2, A_{\mathfrak{P}}))[f,j]$ and define a complex number $\Omega(f,j;A_{\mathfrak{P}})$ by

$$(\delta(f) + jF_\infty(\delta(f)))/2 = \Omega(f,j;A_{\mathfrak{P}})\eta(f,j;A_{\mathfrak{P}}).$$

This $\Omega(f,j;A_{\mathfrak{P}})$ is uniquely determined up to constant multiple of units in $A_{\mathfrak{P}}$. We call $\Omega(f,+;A_{\mathfrak{P}})$ and $\Omega(f,-;A_{\mathfrak{P}})$ the Eichler-Shimura periods. For $j = \pm, 1 \le l \le 2k-n-1$, and a Dirichlet character χ such that $\chi(-1) = j(-1)^{l-1}$, put

$$\mathbf{L}(l,f,\chi) = \mathbf{L}(l,f,\chi;A_{\mathfrak{P}}) = \frac{\Gamma(l)L(l,f,\chi)}{\tau(\chi)(2\pi\sqrt{-1})^l\Omega(f,j;A_{\mathfrak{P}})},$$

where $\tau(\chi)$ is the Gauss sum of χ. In particular, put $\mathbf{L}(l,f;A_{\mathfrak{P}}) = \mathbf{L}(l,f,\chi;\mathfrak{P})$ if χ is the principal character. Furthermore, put

$$\mathbf{L}(l,f,\mathrm{Ad}) = \frac{\widetilde{\Lambda}(l,f,\mathrm{Ad})}{\langle f,f\rangle}.$$

Then it is well-known that $\mathbf{L}(l, f, \chi)$ belongs to the field $K(\chi)$ generated over K by all the values of χ, and $\mathbf{L}(l, f, \mathrm{Ad})$ belongs to $\mathbf{Q}(f)$ (cf. [Bo2], [Sh1].) Let $\hat{f}$ be the Ikeda lift of f. Let $S_k(\Gamma_n)^*$ be the subspace of $S_k(\Gamma_n)$ generated by all the Ikeda lifts $\hat{g}$ of primitive forms $g \in S_{2k-n}(\Gamma_1)$. We remark that $S_k(\Gamma_2)^*$ is the Maass subspace of $S_k(\Gamma_2)$. As for this, the first named author[Kat5] proposed the following conjecture:

Conjecture D. ([Kat5]) *Let K and f be as above. Assume that $k > n$. Let $\mathfrak{P}$ be a prime ideal of K not dividing $(2k-1)!$. Then $\mathfrak{P}$ is a congruence prime of $\hat{f}$ with respect to $(S_k(\Gamma_n)^*)^{\perp}$ if and only if $\mathfrak{P}$ divides $\mathbf{L}(k, f) \prod_{i=1}^{n/2-1} \mathbf{L}(2i+1, f, \mathrm{Ad})$.*

In [Kat5], we explained why our conjecture is reasonable. We review it:

Theorem 5.2. *Let K and f be as above. Assume that the Conjecture A holds for f. Let $\mathfrak{P}$ be a prime ideal of K. Furthermore assume that*
(1) $\mathfrak{P}$ *divides* $\mathbf{L}(k, f) \prod_{i=1}^{n/2-1} \mathbf{L}(2i+1, f, \mathrm{Ad})$.
(2) $\mathfrak{P}$ *does not divide*

$$\tilde{\xi}(2m) \prod_{i=1}^{n} \mathbf{L}(2m+k-i, f) \mathbf{L}(k-n/2, f, \chi_D) D (2k-1)!$$

for some integer $2 \le m \le k/2 - n/2$, and for some fundamental discriminant D such that $(-1)^{n/2} D > 0$.
Then $\mathfrak{P}$ is a congruence prime of $\hat{f}$ with respect to $\mathbf{C}\hat{f}^{\perp}$. Furthermore assume that either one of the following conditions:
(3-1) $n = 2$, *and f is ordinary at the prime number p divided by $\mathfrak{P}$.*
(3-2) *f is ordinary at the prime number p divided by $\mathfrak{P}$, and $\mathfrak{P}$ does not divide*

$$\prod_{p \le (2k-n)/12} (1 + p + \cdots + p^{n-1}) \frac{\langle f, f \rangle}{\Omega(f, +, A_{\mathfrak{P}}) \Omega(f, -, A_{\mathfrak{P}})}.$$

Then $\mathfrak{P}$ is a congruence prime of $\hat{f}$ with respect to $(S_k(\Gamma_n)^*)^{\perp}$.

As stated above, Conjecture A is true in case $n = 2$ and 4. Thus the assertion of Theorem 5.2 holds without assuming Conjecture A in this case. We remark that a similar result has been obtained in case $n = 2$ by Brown[Br].

Key ingredients of the proof of Theorem 5.2 is as follows:
(1) Pullback formula of the Siegel Eisenstein series acted by a certain differential operator(Boecherer[Bo1],[Bo2], Shimura[Sh4])
(2) Characterization of congruence primes of f (Hida[Hi3])
(3) Waldspruger type formula on $\mathbf{L}(k-n/2, f, \chi_D)$ (Kohnen-Zagier[K-Z])

Example Let $n=4$ and $k=18$. Then we have
dim $S_{18}(\Gamma_4) \approx 16$ (cf. Poor and Yuen[P-Y])
and
dim $S_{18}(\Gamma_4)^* =$ dim $S_{32}(\mathit{\Gamma}_1) = 2$.
Take a primitive form $f \in S_{32}(\mathit{\Gamma}_1)$. Then we have $[\mathbf{Q}(f):\mathbf{Q}] = 2$, and $211 = \mathfrak{P}\mathfrak{P}'$ in $\mathbf{Q}(f)$. Then we have

$$N_{\mathbf{Q}(f)/\mathbf{Q}}(\mathbf{L}(18, f)) = 2^7 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 211,$$

$$N_{\mathbf{Q}(f)/\mathbf{Q}}(\prod_{i=1}^{4} \mathbf{L}(24-i, f)) = 2^{19} \cdot 3^{13} \cdot 5^5 \cdot 7^8 \cdot 11^2 \cdot 13^5 \cdot 17^5 \cdot 19^3 \cdot 23 \cdot 503 \cdot 1307 \cdot 14243,$$

$$\tilde{\xi}(6) = 2^{-2} \cdot 3^{-2} \cdot 7^{-1}$$

and

$$N_{\mathbf{Q}(f)/\mathbf{Q}}(\mathbf{L}(16, f, \chi_1) = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13^2.$$

(cf. Stein [St].) Furthermore, by a direct computation we see neither $\mathfrak{P}$ nor $\mathfrak{P}'$ is a congruence prime of $\hat{f}$ with respect to $\mathbf{C}\hat{g}$ for another primitive form $g \in S_{32}(\mathit{\Gamma}_1)$. Thus by Theorem 5.2, $\mathfrak{P}$ or $\mathfrak{P}'$ is a congruence prime of $\hat{f}$ with respect to $S_{18}(\Gamma_4)^{*\perp}$.

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