Spinor L-functions of Arakawa lifting

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0 Introduction.

The purpose of this note is to give a brief review on our result [M-N, Theorem 5.1, Corollary 5.2] of a theta lifting from automorphic forms on the inner form $O^*(4)$ of the othogonal group of degree four (realized as the quaternion unitary group attached to the quaternion skew-Hermitian form of degree two) to automorphic forms on the quaternion unitary group $Sp(1,1)$ of signature $(1+,1-)$. The lifting in our concern was originally considered by Tsuneo Arakawa in the non-adelic setting. We thus call it *Arakawa lifting.* We give a formula for all the non-Archimedean local factors of the spinor L-function of the lift.

In [Ar] Arakawa formulated a theta lifting from elliptic cusp forms to automorphic forms on $Sp(1,q)_{\mathbb{R}}$, which was inspired by Kudla lifting (cf. [Ku]), i.e. a theta lifting from elliptic cusp forms to holomorphic automorphic forms on $SU(1,q)_{\mathbb{R}}$. After Shintani [Shin] Arakawa constructed a theta kernel function on $\mathfrak{h}\times SO(4,4q)_{\mathbb{R}}$ by the Weil representation and considered its restriction to $\mathfrak{h}\times Sp(1,q)_{\mathbb{R}}$ to formulate the lift, where $\mathfrak{h}$ denotes the complex upper half plane. In [Ar] and [N] the images of the lift are proved to be automorphic forms generating quaternionic discrete series in the sense of Gross-Wallach [G-W].

On the other hand, (when $q=1$) Arakawa's theta lift can be naturally reformulated as a theta lifting from an automorphic form (f,f') on $GL_{2,\mathbb{A}}\times B_{\mathbb{A}}^{\times}$ to a form on $GSp(1,1)_{\mathbb{A}}$, where $B^{\times}$ denotes the multiplicative group of a definite quaternion algebra B over $\mathbb{Q}$ and f an elliptic cusp form. At the Archimedean place our lift is a theta correspondence between a discrete series of $O^*(4)_{\mathbb{R}}\simeq SL_2(\mathbb{R})\times SU(2)_{\mathbb{R}}$ and a quaternionic discrete series of $Sp(1,1)_{\mathbb{R}}$. Namely Arakawa lifting can be understood in the framework of the theta correspondence for dual reductive pairs; the pair $O^*(4)\times Sp(1,1)$ is the dual reductive pair for the lift.

1 Formulation of the lift.

Throughout this note we fix a definite quaternion algebra B over $\mathbb{Q}$ and its maximal order $\mathcal{O}$, and denote by $\mathcal{G}_{\mathbb{A}}$ the adelization of an algebraic group $\mathcal{G}$ over $\mathbb{Q}$.

We define a $\mathbb{Q}$-algebraic group $G=GSp(1,1)$ by

$$G_{\mathbb{Q}}:=\{g\in M_2(B)\mid {}^t\bar{g}\begin{pmatrix}0&1\\1&0\end{pmatrix}g=\nu(g)\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \nu(g)\in\mathbb{G}_m\}.$$

We denote by d_B the discriminant of B. For each finite prime p we set

$$K_p^1 := \{k \in G_{\mathbb{Q}_p} \mid k(\mathcal{O}_p^{\oplus 2}) = (\mathcal{O}_p^{\oplus 2})\},$$

where $\mathcal{O}_p$ stands for the p-adic completion of $\mathcal{O}$. For a prime divisor p of d_B we put

$$K_p^2 := \{k \in G_{\mathbb{Q}_p} \mid k(\mathcal{O}_p \oplus \mathfrak{P}_p^{-1}) = (\mathcal{O}_p \oplus \mathfrak{P}_p^{-1})\},$$

where $\mathfrak{P}_p$ denotes the maximal ideal of $\mathcal{O}_p$. These K_p^1 and K_p^2 form maximal compact subgroups of $G_{\mathbb{Q}_p}$. For a divisor D of d_B we set

$$K_f^D := \prod_{p \nmid d_B} K_p^1 \prod_{p \mid D} K_p^1 \prod_{p \mid \frac{d_B}{D}} K_p^2.$$

We furthermore put

$$G_{\mathbb{R}}^1 := \{g \in G_{\mathbb{R}} \mid \nu(g) = 1\},$$
$$K_\infty := G_{\mathbb{R}}^1 \cap \{g \in M_2(\mathbb{H}) \mid {}^t\bar{g}g = 1_2\},$$

where $\mathbb{H} \simeq B \otimes_{\mathbb{Q}} \mathbb{R}$ denotes the Hamilton quaternion algebra. This K_∞ is a maximal compact subgroup of $G_{\mathbb{R}}^1$. We can identify K_∞ with two copies of the special unitary group $SU(2)$ of degree two. For a positive integer κ we let (τ_κ, V_κ) be an irreducible represerntation of K_∞ given by

$$\tau_\kappa := \mathrm{id}_{SU(2)} \boxtimes \sigma_\kappa,$$

where $(\sigma_\kappa, V_\kappa)$ denotes the κ-th symmetric tensor representation of $SU(2)$.

Definition 1.1. Let $\kappa > 4$ be an even integer. We denote by $\mathcal{S}_\kappa^D$ the space of bounded V_κ-valued smooth functions F on $G_{\mathbb{A}}$ satisfying the following two conditions:

1. $F(z\gamma g k_f k_\infty) = \tau_\kappa(k_\infty)^{-1} F(g), \quad \forall (z, \gamma, g, k_f, k_\infty) \in Z_{G_{\mathbb{A}}} \times G_{\mathbb{Q}} \times G_{\mathbb{A}} \times K_f^D \times K_\infty,$
2. For each fixed $g_0 \in G_{\mathbb{A}}$, the right translations of coefficients of $F(g_0 *)|_{G_{\mathbb{R}}^1}$ (regarded as functions on $G_{\mathbb{R}}^1$) by $G_{\mathbb{R}}^1$ genarate the discrete series representation with minimal K_∞-type τ_κ (quaternionic discrete series).

Here $Z_{G_{\mathbb{A}}}$ denotes the center of $G_{\mathbb{A}}$.

Let $S_\kappa(D)$ be the space of elliptic cusp forms of weight κ with respect to the congruence subgroup $\Gamma_0(D)$ (regarded as automorphic forms on $GL_{2,\mathbb{A}}$). We furthermore let $\mathcal{A}_\kappa$ be the space of automorphic forms on $B_{\mathbb{A}}^\times$ of weight σ_κ which are left $\prod_{p<\infty} \mathcal{O}_p^\times$-invariant, where we regard σ_κ as a representation of $\mathbb{H}^1 \simeq SU(2)$. Both forms are assumed to have the trivial central character.

Let r be the metaplectic representation of $G_{\mathbb{A}} \times GL_{2,\mathbb{A}} \times B_{\mathbb{A}}^{\times}$ introduced in [M-N, §3, §4]. This is realized on the Schwartz-Bruhat space $\mathcal{S}(B_{\mathbb{A}}^{\oplus 2} \times \mathbb{Q}_{\mathbb{A}}^{\times})$ on $B_{\mathbb{A}}^{\oplus 2} \times \mathbb{Q}_{\mathbb{A}}^{\times}$ whose Archimedean component is the space of "End(V_κ)-valued Schwartz functions" on $\mathbb{H}^{\oplus 2} \times \mathbb{R}^{\times}$ in some modified sense. At each finite place p we set

$$\varphi_p(X,t) := \begin{cases} \mathrm{char}_{\mathcal{O}_p \oplus \mathfrak{P}_p^{-1}}(X)\,\mathrm{char}_{\mathbb{Z}_p^{\times}}(t) & (p|\frac{d_B}{D}) \\ \mathrm{char}_{\mathcal{O}_p^{\oplus 2}}\,\mathrm{char}_{\mathbb{Z}_p^{\times}}(t) & (p : \text{otherwise}) \end{cases}$$

and at the infinite place

$$\varphi_\infty(X,t) := \begin{cases} t^{\frac{\kappa+3}{2}} \sigma_\kappa(X_1 + X_2) \exp(-2t\pi {}^t\bar{X}X) & (t > 0) \\ 0 & (t < 0) \end{cases},$$

where $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ (resp. t) is the variable for $B_v^{\oplus 2}$ (resp. $\mathbb{Q}_v^{\times}$) with the completion of $B^{\oplus 2}$ (resp. $\mathbb{Q}^{\times}$) at a place $v \le \infty$. With the test function $\varphi := \prod_{v \le \infty} \varphi_v \in \mathcal{S}(B_{\mathbb{A}}^{\oplus} \times \mathbb{Q}_{\mathbb{A}}^{\times})$ we define an End(V_κ)-valued theta series $\Theta(g,h,h')$ on $G_{\mathbb{A}} \times GL_{2,\mathbb{A}} \times B_{\mathbb{A}}^{\times}$ by

$$\theta(g,h,h') := \sum_{(X,t) \in B^{\oplus 2} \times \mathbb{Q}^{\times}} r(g,h,h')\varphi(X,t).$$

For $\kappa > 6$ our theta lift is then defined as follows:

$$S_\kappa(D) \times \mathcal{A}_\kappa \ni (f,f') \mapsto \int_{Z_{1,\mathbb{A}} GL_{2,\mathbb{Q}} \backslash GL_{2,\mathbb{A}}} dh \int_{Z_{2,\mathbb{A}} B^{\times} \backslash B_{\mathbb{A}}^{\times}} dh' \overline{f(h)} \Theta(g,h,h') f'(h') \in \mathcal{S}_\kappa^D.$$

Here $Z_{1,\mathbb{A}}$ and $Z_{2,\mathbb{A}}$ stand for the center of $GL_{2,\mathbb{A}}$ and $B_{\mathbb{A}}^{\times}$ respectively. We denote this by $\mathcal{L}(f,f')(g)$.

2 Statement of the result.

Let $F \in \mathcal{S}_\kappa^D$ be a Hecke eigenform. For such F we denote by $L(F, spin, s)$ its spinor L-function. We define the local factor $L_p(F, spin, s)$ at an unramified finite prime p (resp. ramified finite prime) using the formula for the denominator of the Hecke series by Shimura [Shim Theorem 2] (resp. Hina-Sugano [H-S §4]). Now we state our main result ([M-N, Theorem 5.1, Corollary 5.2]).

Theorem 2.1. *Assume that $(f,f') \in S_\kappa(D) \times \mathcal{A}_\kappa$ are Hecke eigenforms.*
(1) The theta lift $\mathcal{L}(f,f')(g)$ is also a Hecke eigenform and its eigenvalues are given explicitly in terms of those of (f,f').
(2) The spinor L-function $L(F, Spin, s)$ for $F = \mathcal{L}(f,f')$ is written as

$$L(\mathcal{L}(f,f'), spin, s) = L^D(f,s)\, L^{d_B}(f',s) \prod_{p|D} \left(1 - \{\lambda_p + (1-p)\lambda_p'\} p^{\kappa-2-s} + p^{2\kappa-3-2s}\right)^{-1},$$

where λ_p *(resp.* λ'_p*) is a Hecke eigenvalue of* f *(resp.* f'*). In particular, when* $D = 1$*, we have*

$$L(\mathcal{L}(f, f'), spin, s) = L(f, s)\, L^{d_B}(f', s),$$

where, for a finite set S *of non-Archimedean primes,* $L^S(*, s)$ *denotes the partial L-function whose local factors run over the non-Archimedean primes outside* S*.*

Our approach to get this theorem is to calculate the commutation relations of Hecke operators for the local metaplectic representation of $GSp(1,1)_{\mathbb{Q}_p} \times GL_{2,\mathbb{Q}_p} \times B^\times_{\mathbb{Q}_p}$. Such calculation leads to the result on the Hecke eigenvalues. The formula for the L-function is an immediate consequence of this.

Remark 2.2. For $p|D$ let ϵ_p and ϵ'_p be the signature of the Atkin-Lehner involution for an elliptic cusp form f and an automorphic form f' on $B^\times_{\mathbb{A}}$ respectively. By a direct computation of the local metaplectic representation we see that

$$\mathcal{L}(f, f') \not\equiv 0 \Rightarrow \epsilon_p = \epsilon'_p \quad \forall p|D.$$

References

[Ar] T. Arakawa, unpublished note.

[G-W] B. Gross and N. Wallach, On quaternionic discrete series representations, and their continuations, J. Reine. Angew. Math., 481 (1996), 73-123.

[H-S] T. Hina and T. Sugano, On the local Hecke series of some classical groups over $\mathfrak{p}$-adic fields, J. Math. Soc. Japan, 35 (1983), 133-152.

[Ku] S. Kudla, On certain arithmetic automorphic forms for $SU(1, q)$, Invent. Math., 52 (1979), 1-25.

[M-N] A. Murase and H. Narita, Commutation relation of Hecke operators for Arakawa lifting, MPI-preprint, MPIM2006-25.

[N] H. Narita, Theta lifting from elliptic cusp forms to automorphic forms on $Sp(1, q)$, preprint, (2006).

[Shim] G. Shimura, On modular correspondence for $Sp(N, \mathbb{Z})$ and their congruence relations, Proc. Nat. Acad. Sci. U.S.A., 49 (1963), 824-828.

[Shin] T.Shintani, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J. 58, (1975), 83-126.

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