

Proof of van Geemen-Nygaard-van Straten's conjectures on Siegel modular threefold varieties.

TAKEO OKAZAKI

Department of Mathematics Graduate School of Science Osaka University,
E-mail: okazaki@gaia.math.wani.osaka-u.ac.jp

1 Introduction and the Conjectures.

As a next step of the Eichler-Shimura theory, the $GSp(4, \mathbb{Q})$ case is studied by some researchers. But, there are still problems in the relation between the L -function of modular varieties and modular forms.

van Geemen, Nygaard and van Straten in [4], [3] determined the defining equation of some Siegel threefold varieties, and determined their Hasse-Weil zeta functions. Further they gave some conjectures on their zeta functions and the spinor L -functions of Siegel cuspforms of weight 3 on each varieties, which are interpreted as differential 3-forms. We will review their results, quickly.

Let

$$\begin{aligned}\Gamma(4, 8) &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma(4) \mid \text{diag}(B) \equiv \text{diag}(C) \equiv 0 \pmod{8} \right\}, \\ \Gamma(2, 4, 8) &= \left\{ \begin{pmatrix} 1+4A' & B \\ C & 1+4D' \end{pmatrix} \in \Gamma(4, 8) \mid \text{trace}(A') \equiv 0 \pmod{2} \right\}.\end{aligned}$$

Let $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$, $x \in \mathbb{C}$, and $Z[v] = {}^t v Z v$. Let θ_m be the Igusa theta constant associated to a characteristic $m = (m_1, m_2, m_3, m_4)$, that is,

$$\theta_m(Z) = \sum_{a, b \in \mathbb{Z}} \mathbf{e} \left(\frac{1}{2} \left(Z \begin{bmatrix} m_1 + (a/2) \\ m_2 + (b/2) \end{bmatrix} + m_3(m_1 + 2a)/2 + m_4(m_2 + 2b)/2 \right) \right),$$

and let $\Theta_{mn}(Z) = \theta_{(m, n, 0, 0)}(2Z)$. From the theta embedding $\Theta_{(2, 4, 8)} : \Gamma(2, 4, 8) \backslash \mathfrak{H}_2 \longrightarrow \mathbf{P}^{13}$ given by

$$Z \mapsto \overbrace{(\theta_{(0, 0, 0, 0)}(Z), \dots, \theta_{(1, 1, 1, 1)}(Z))}^{\text{ten even theta}}, \Theta_{00}(Z), \Theta_{10}(Z), \Theta_{01}(Z), \Theta_{11}(Z),$$

they derived the defining equation of $A(2, 4, 8) = \Gamma(2, 4, 8) \backslash \mathfrak{H}_2$. Moreover they showed that the image of $\Theta_{(2, 4, 8)}$ is a complete intersection. Using these facts, they concluded that $S_3(\Gamma(2, 4, 8))$ is spanned by the six-tuple products of theta functions in the table below.

space	dim	theta series
$S_3(\Gamma(4))$	15	$\theta_1 = \theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(1,1,0,0)}\theta_{(1,0,0,1)}\theta_{(0,1,1,0)}\theta_{(1,1,1,1)}(\mathbb{Z})$
$S_3(\Gamma(4, 8))$	90	$\theta_2 = \theta_{(0,0,0,0)}\theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}(\mathbb{Z})$
	90	$\theta_3 = \theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}\theta_{(0,0,0,1)}(\mathbb{Z})$
	360	$\theta_4 = \theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}\theta_{(0,0,0,1)}\theta_{(1,0,0,1)}(\mathbb{Z})$
	180	$\theta_5 = \theta_{(0,0,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}\theta_{(0,0,1,1)}\theta_{(0,1,1,0)}\theta_{(1,1,1,1)}(\mathbb{Z})$
	60	$\theta_6 = \theta_{(0,0,0,0)}\theta_{(0,0,0,0)}\theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,0,1,1)}\theta_{(0,1,1,0)}(\mathbb{Z})$
	360	$\theta_7 = \theta_{(0,0,0,0)}\theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(0,0,0,1)}\theta_{(0,0,1,1)}(\mathbb{Z})$
$S_3(\Gamma(2, 4, 8))$	240	$\theta_8 = \Theta_{(0,0)}\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}(\mathbb{Z})$
	360	$\theta_9 = \Theta_{(0,0)}\theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}(\mathbb{Z})$
	288	$\theta_{10} = \Theta_{(0,0)}\theta_{(0,0,0,0)}\theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(0,0,0,1)}\theta_{(0,0,1,1)}(\mathbb{Z})$
	240	$\theta_{11} = \Theta_{(0,0)}\Theta_{(1,0)}\Theta_{(0,1)}\theta_{(0,0,1,0)}\theta_{(0,0,0,1)}\theta_{(0,0,1,1)}(\mathbb{Z})$

Each $Sp_2(\mathbb{Z})$ translation of θ_j spans a finite dimensional subspace of $S_3(\Gamma(2, 4, 8))$.
It holds

$$\left(\sum_{\gamma \in Sp_2(\mathbb{Z})} \mathbb{C} \cdot \theta_i | \gamma\right) \cap \left(\sum_{\gamma \in Sp_2(\mathbb{Z})} \mathbb{C} \cdot \theta_j | \gamma\right) = 0 \quad (i \neq j),$$

and, for example

$$\dim_{\mathbb{C}} \sum_{\gamma \in Sp_2(\mathbb{Z})} \mathbb{C} \cdot \theta_1 | \gamma = 15.$$

The meanings of the numbers in the above table are like this. In particular,

$$\begin{aligned} \dim S_3(\Gamma(4)) &= 15, \dim S_3(\Gamma(4, 8)) = 15 + 90 + \cdots + 360 = 1155, \\ \dim S_3(\Gamma(2, 4, 8)) &= 1155 + 240 + \cdots + 240 = 2283. \end{aligned}$$

Using Igusa's transformation formula for theta function, we can obtain the characters ξ_θ on $\Gamma(2)$ from $\theta \in S_3(\Gamma(2, 4, 8))$ by:

$$\xi_\theta(\gamma) = \frac{\theta | \gamma}{\theta}.$$

It holds $\xi_\theta^4 = 1$.

They showed that ξ_θ is characterized by a unique θ , and determined all Hecke eigenforms in $S_3(\Gamma(2, 4, 8))$ in the following way.

When $\xi_\theta^2 = 1$, the Hecke algebra $\mathcal{H}_{(\tilde{2})} = \otimes_{p \neq 2} \mathcal{H}_v(GSp_2(\mathbb{Q}_p), GSp_2(\mathbb{Z}_p))$ outside of 2 acts on the one dimensional space $\mathbb{C}\theta$, and thus θ is a Hecke eigenform. When ξ_θ is not real-valued, $\mathcal{H}_{(\tilde{2})}$ acts on the two dimensional space spanned by θ and θ' , where $\xi_{\theta'} = \overline{\xi_\theta}$. Thus, an appropriate linear combination of θ and θ' is a Hecke eigenform (c.f. Proposition 7.4. of [4]).

Computing some Hecke operators on these eigenforms, they gave

CONJECTURE (VAN GEEMEN-NYGAARD-VAN STRATEN [3], [4]) *Their spinor L-functions are described as follows.*

label	eigenform	conjectured spinor L -function outside of 2
R_6^-	$F_1 = \theta_1$	$\zeta(s-1)\zeta(s-2)L(s, \rho_1)$
$R_4^-(0; 2)$	$F_2 = \theta_2 - 4\theta'_2$	$\zeta(s-1)\zeta(s-2)L(s, \rho_1)$
$R_4(1, 1; 0)$	$F_3 = \theta_3 + 16\theta'_3$	$\zeta(s-1)\zeta(s-2)L(s, \rho_1 \otimes \omega_{-1})$
$R_4^-(1; 1)$	$F_4 = \theta_4 + 4\theta'_4$	$L(s-1, \theta_\mu \otimes \omega_{-2})L(s, \rho_3 \otimes \omega_{-2})$
R_6^*	$F_5 = \theta_5$	$L(s-1, \theta_\mu)L(s, \rho_2)$
$R_4^-(2; 0)$	$F_6 = \theta_6$	$L(s-1, \theta_\mu \otimes \omega_{-2})L(s, \rho_2 \otimes \omega_{-2})$
$R_5^*(1; 0)$	$F_7 = \theta_7$	$L(s, \theta_\mu \otimes \psi_1)$
$R_4^-(1; 0)$	$F_8 = \theta_8$	$L(s, \lambda)$
$R_4^-(0; 1)$	$F_9 = \theta_9$	some π on $SL_2(\mathbb{C})$?
R_5^*	$F_{10} = \theta_{10}$	ω_{-2} -twist of $L(s, \theta_\mu \otimes \psi_1)$
$R(3; 3)$	$F_{11} = \theta_{11}$	$L(s-1, \omega_{-2})L(s-2, \omega_{-2})L(s, \rho_1 \otimes \omega_{-2})$

Here ω_d denotes the quadratic character associated to the extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. \otimes denotes the convolution product. $\theta_\mu, \rho_i, \psi_1$ denote some elliptic eigenforms belonging to the spaces as follows.

elliptic cuspform	space
θ_μ	$S_2(\Gamma_0(32))$
ψ_1	$S_3(\Gamma_0(32), \omega_{-1})$
ρ_1	$S_4(\Gamma_0(8))$
$\rho_2 = \theta_\mu^3$	$S_4(\Gamma_0(32))$
ρ_3	$S_4(\Gamma_0(32))$

and λ is a Größencharacter of $\mathbb{Q}(\mathbf{e}(\frac{1}{8}))$. □

In particular, θ_μ is obtained by the Größen-character μ

$$\theta_\mu(z) = \sum_{\mathfrak{a}} \mu(\mathfrak{a}) \mathbf{e}(N(\mathfrak{a})z), \quad z \in \mathfrak{H}$$

where \mathfrak{a} runs through all integral ideals of $\mathbb{Z}[i]$ prime to 2. μ is related to the elliptic curve $E : y^2 = x^3 - x$ with complex multiplication. (By the way, an explicit birational map between a certain threefold and E are given, related to the Hodge-Tate conjecture.)

For these conjectures, our main result is

MAIN THEOREM *The conjectures for $F_i, 1 \leq i \leq 6$ are true.*

Our proof is done by the Yoshida lift as follows. The conjectured $L(s, F_i)$ for $1 \leq i \leq 6$ are products of L -functions of elliptic modular forms, and the Yoshida lift [13] can provide a Siegel modular form having such type of L -functions. Indeed, in each spaces $S_3(\Gamma(2), \xi_{F_i})$ (dimension one or two) containing F_i , we can construct Hecke eigenforms having the conjectured L -function, by the Yoshida lift (modified slightly in [7]). For example,

$$\begin{aligned} (\theta_\mu, \rho_2) &\mapsto Y_{\theta_\mu, \rho_2} \in S_3(\Gamma(2), \xi_{F_5}) = \mathbb{C} \cdot F_5, \\ L(s, Y_{\theta_\mu, \rho_2}, \text{spin}) &= L(s-1, \theta_\mu)L(s, \rho_2), \end{aligned}$$

thus $L(s, F_5)$ must be $L(s-1, \theta_\mu)L(s, \rho_2)$ as conjectured.

REMARK 1.1 $\zeta(s)\zeta(s-1)$ or its twists are contained in the L -functions of F_1, F_2, F_3, F_{11} . They are L -functions of elliptic Eisenstein series. (c.f. Saito-Kurokawa lift.)

REMARK 1.2 $L(s, \lambda)$ is considered as a L -function of a Hilbert modular form over $\mathbb{Q}(\sqrt{2})$ of weight $(2, 4)$. Certainly, the Yoshida lift of another version (studied deeply by B. Roberts) maps Hilbert modular forms to Siegel modular form of weight 3, our technique will work for the conjecture on F_8 . But we have not tried.

2 Proof for F_5 .

We give a quick review of the Yoshida lift which was modified in our previous work [7]. Let

$$D_{\mathbb{Q}} = \mathbb{Q} + \mathbb{Q}I + \mathbb{Q}J + \mathbb{Q}IJ, I^2 = -1, J^2 = -1, IJ = -JI.$$

and R be an order in $D_{\mathbb{Q}}$. Let φ_1 and φ_2 are eigenforms on $D_{\mathbb{A}}^{\times}$. Then, by Jacquet-Langlands theory, their L -functions are equal to those of elliptic modular forms. Here φ_i are called automorphic forms, if

$$(2.1) \quad \varphi_i(\gamma x k) = \xi(k)\varphi_i(x),$$

for $\gamma \in D_{\mathbb{Q}}^{\times}, x \in D_{\mathbb{A}}^{\times}, k \in R_p^{\times}$. Here $\xi = \otimes_p \xi_p$ is a character on $\otimes_p R_p^{\times}$. Associated to the pair φ_1, φ_2 , we take a certain Schwartz function (i.e., theta kernel or test function) $\Phi = \prod_v \Phi_v$ on $D_{\mathbb{A}}^2$ so that

i) If ξ_p on R_p^{\times} is trivial, Φ_p is the characteristic function of R_p^2 .

ii) If ξ_p is non-trivial, Φ_p has the property such as

$$(2.2) \quad \Phi_p(k_1^{-1}x_1k_2, k_1^{-1}x_2k_2) = \xi_p(k_1^{-1}k_2)\Phi_p(x_1, x_2),$$

for $k_i \in R_p^{\times}, x_j \in D_p$

Further, for our purpose, assume that φ_1 is corresponding (by Jacquet-Langlands) to elliptic modular form of weight 2 (see Böcherer and Schulze-Pillot's works on the Yoshida lift for general case). Then, we obtain a Siegel modular form of degree 2 of weight 3.

THEOREM 2.1 (THE YOSHIDA LIFT.) Set a function of $Z \in \mathfrak{H}_2$ by

$$Y_{\varphi_1 \times \varphi_2, \Phi}(Z) := \sum_{i,j=1}^h (n_i n_j)^{-1} \sum_{x_1, x_2 \in D_{\mathbb{Q}}} \Phi_0(y_i^{-1}x_1y_j, y_i^{-1}x_2y_j) P_j(x_1^*x_2) \varphi_1(y_i) \mathbf{e}(\mathrm{tr}(\begin{pmatrix} n(x_1) & \mathrm{tr}(x_1x_2^*) \\ \mathrm{tr}(x_1x_2^*) & n(x_2) \end{pmatrix} Z))).$$

Here $n_i = \#(D_{\mathbb{Q}} \cap y_i^{-1}R_{\mathbb{A}}y_i)$ for the decomposition

$$D_{\mathbb{A}}^{\times} = \bigsqcup_{i=1}^h D_{\mathbb{Q}}^{\times} y_i R_{\mathbb{A}}^{\times},$$

Φ_0 is the finite part of Schwartz function: $\prod_{p < \infty} \Phi_p$. P_j are certain pluri-harmonic polynomials on \mathbb{H}^2 obtained from the information of φ_2 . Then, $Y_{\varphi_1 \times \varphi_2, \Phi}$ is a Siegel modular cuspform of weight 3 and

$$L(s, Y_{\varphi_1 \times \varphi_2, \Phi}, \mathrm{spin}) = L(s-1, \varphi_1)L(s, \varphi_2).$$

□

On the other hand, since the ten tuple product (denoted by χ_5 in many references) of even theta functions are in $S_5(\Gamma(2))$ and $F_5 = \theta_5$ is a distinct six tuple product, we find ξ_{F_5} on $\Gamma(2)$ is equal to the character associated to the complement four tuple product θ_5^\vee of even θ 's:

$$\begin{aligned}\theta_5^\vee &:= \theta_{(1,0,0,0)}\theta_{(0,1,0,0)}\theta_{(1,1,0,0)}\theta_{(1,0,0,1)}(Z), \\ \xi_{F_5}\xi_{\theta_5^\vee} &= \xi_{\chi_5} = 1 \quad \text{on } \Gamma(2).\end{aligned}$$

This four tuple product of theta function is related to the quadratic form of rank 4. So, we can identified the quadratic form with the norm form of the above quaternion algebra $D_{\mathbb{Q}}$! This identification gives us a theta kernel for the modified Yoshida lift, such as

$$\Phi_2(x_1, x_2) = \begin{cases} e(d_2/4) & \text{if } x_1 = a_1 + b_1I + c_1J + d_1IJ \equiv 1 + J + IJ, \\ & \text{and } x_2 = a_2 + b_2I + c_2J + d_2IJ \equiv I + J \pmod{2}, \\ 0 & \text{otherwise.} \end{cases}$$

(see Introduction and Main idea of [7] for more explanations.)

Then, from (2.2) we get a character $\xi = (\xi)_2 \times \prod_{v \neq 2} 1_v$ on $R_{\mathbb{A}}^\times$ such as

$$\begin{aligned}R &= \mathbb{Z} + 2\mathbb{Z}I + 2\mathbb{Z}J + 2\mathbb{Z}IJ, \\ \xi_2(k) &= (1 + 2a + 2bI + 2cJ + 2dIJ) = (-1)^{b+c}.\end{aligned}$$

We calculate the class number of $D_{\mathbb{Q}}$ related to R is 2, so there exist only 2 automorphic forms satisfying (2.1), essentially. Among them, we select the eigenforms φ_1, φ_2 having the same L -function of elliptic modular forms $\theta_{\mu^3}, \theta_{\mu}$, and check that the Fourier coefficient at $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ of $Y_{\varphi_1 \times \varphi_2, \Phi}$ is not zero. Thus, certainly there exists Hecke eigenform in $S_3(\Gamma(2), \xi_{F_5})$ having the conjectured spinor L -function, which finishes the proof.

By the way, van Geemen and Nygaard [3] determined the defining equation of the Siegel threefold $V_5 = \ker(\xi_{F_5}) \backslash \mathfrak{H}_2$, on which F_5 rides as a holomorphic differential form (other cuspforms in $S_3(\Gamma(2, 4, 8))$ don't ride on). They showed

$$h^3(V_5, \mathbb{Q}_l) = 4, \quad L(s, H^3(V_5, \mathbb{Q}_l)) = L(s-1, \mu)L(s, \rho_2).$$

So, for the threefold variety V_5 , we have the coincidence of L -function of Galois representation and that of cuspform.

3 For F_7 .

The conjectured L -function of F_7 is a convolution product of L -function of elliptic modular forms. Because such type of lift from elliptic modular forms cannot obtain by the Yoshida lift, the similar technique for F_5 doesn't work. Although we are in preparation, we will describe the sketch of our idea for F_7 .

- 1). Construct the threefold variety V_7 which only F_7 rides on. V_7 will be given by a certain projection of $A(2, 4, 8)$.
- 2). Show $h^3(V_7, \mathbb{Q}_l) = 4$. It is possible by calculating the Euler number of V_7 and counting some numbers of \mathbb{F}_p -rational points of the p reduction of V_7 .

- 3). Show that the Galois representation ρ_τ associated to $H^3(V_7, \mathbb{Q}_l)$ has the same zeta function conjectured: $L(s, \rho_\tau) = L(s, \psi \otimes \theta_\mu)$.
- 4). By the congruence relation, we obtain $L(s, F_7, spin) = L(s, \rho_\tau) = L(s, \psi \otimes \theta_\mu)$.

We will only explain (perhaps) the most remarkable technique in 3).

THEOREM 3.1 (OKAZAKI) *Suppose that $\rho, \rho' : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GSp_2(\mathbb{Z}_2)$ is ramified at 2 and ∞ , at most. If $L(s, \rho)_p = L(s, \rho')_p$ for every p in a certain finite set Σ of rational primes, then it holds $L(s, \rho) = L(s, \rho')$. \square*

In this theorem, Σ is determined by a ‘finite’ non-fifth degree set $T \subset Im(\overline{\rho}(Gal(\overline{\mathbb{Q}}/\mathbb{Q}))) \subset (GSp_2(\mathbb{Z}_2))^{ab}/((GSp_2(\mathbb{Z}_2))^{ab})^2$ below, so that

$$\overline{\rho}(\mathfrak{F}_p) \supset T, \quad p \in \Sigma.$$

Here $(GSp_2(\mathbb{Z}_2))^{ab}$ stands for the abelianization, and $\overline{\rho}$ means the projection of ρ to $(GSp_2(\mathbb{Z}_2))^{ab}/((GSp_2(\mathbb{Z}_2))^{ab})^2$. Notice that $(GSp_2(\mathbb{Z}_2))^{ab}/((GSp_2(\mathbb{Z}_2))^{ab})^2$ is a \mathbb{F}_2 -vector space.

DEFINITION 3.2 (NON-FIFTH DEGREE SET.) *Let W be a finite dimensional vector space over \mathbb{F}_2 . If every homogeneous polynomial of degree five on W which vanishes on T vanishes on W , we say T is a non-fifth degree set.*

This theorem is obtained by the following two facts.

- $M/2M$ is spanned over \mathbb{F}_2 by $(\overline{\rho(\mathfrak{F}_p)}, \overline{\rho'(\mathfrak{F}_p)})$, $p \in \Sigma$, where $M = \mathbb{Z}_2(\rho(\mathfrak{F}_p), \rho'(\mathfrak{F}_p)) \subset M_{4,8}(\mathbb{Z}_2)$.
- By Nakayma’s lemma, M is spanned over \mathbb{Z}_2 by $(\rho(\mathfrak{F}_p), \rho'(\mathfrak{F}_p))$, $p \in \Sigma$. So we conclude

$$(3.3) \quad \sigma_j(\rho(\mathfrak{F}_p)) - \sigma_j(\rho'(\mathfrak{F}_p)) = 0,$$

if (3.3) holds for $\forall p \in \Sigma$. Here σ_j means the elementary symmetric polynomials of 4×4 sized matrix. Thus, at every p ,

$$L(s, \rho)_p = L(s, \rho')_p,$$

Since the characteristic polynomials of $\rho(\mathfrak{F}_p), \rho'(\mathfrak{F}_p)$ is determined by σ_j .

REMARK 3.3 *This result is analogous to those for $GL(2)$ -representation by Livné [5] and Tate [12].*

REMARK 3.4 *Poor and Yuen’s technique in [1] provides good estimations for the dimensions of $S_k^n(Sp_n(\mathbb{Z}))$ and $S_k^n(\Gamma)$ for $\Gamma \subset Sp_n(\mathbb{Z})$, and determined some graded rings of some Siegel modular forms of degree n .*

Their technique says that two Siegel cuspforms coincides, if sufficiently many Fourier coefficients of them coincide. However, our theorem 3.1 may imply a remarkable progress in the estimation of the dimensions.

REMARK 3.5 *To say the coincidence of the spinor L -function and the Galois representation by the congruence relation, we need the irreducibility of the Galois representation. Since the Galois representation associated to $H^3(V_5, \mathbb{Q}_l)$ is reducible, we cannot conclude that $L(s, F_5, spin) = L(s - 1, \theta_\mu)L(s, \rho_2)$, only from their result $L(s, H^3(V_5, \mathbb{Q}_l)) = L(s - 1, \theta_\mu)L(s, \rho_2)$.*

References

- [1] C. Poor, D. Yuen: Linear dependence among Siegel modular forms, *Math. Ann.* **312** (2000), 205-234.
- [2] G. Faltings and C-L, Chai: *Degeneration of Abelian varieties*, Springer (1990).
- [3] B. van Geemen, N. O. Nygaard: On the Geometry and Arithmetic of Some Siegel Modular Threefolds, *J. Number Theory.* **53** (1995), 45-87.
- [4] B. van Geemen, D. van Straten: The cuspform of weight 3 on $\Gamma_2(2, 4, 8)$, *Math. Comp.* **61** (1993), 849-872.
- [5] R. Livné: Cubic exponential sums and Galois representations. *Contemp. Math.* **67** (1987), 247-261.
- [6] J. Igusa: *Theta functions*, Springer-Verlag (1972).
- [7] T. Okazaki: Proof of R. Salvati Manni and J. Top's conjectures on Siegel modular forms and Abelian surfaces, *Amer. J. Math.* **128** (2006), 139-165.
- [8] —: L -functions of $S_3(\Gamma(4, 8))$, to appear in *J. Number Theory*.
- [9] R. Salvati Manni, J. Top: Cuspforms of weight 2 for the group $\Gamma(4, 8)$, *Amer. J. Math* **115** (1993), 455-486.
- [10] G. Shimura: *Introduction to the arithmetic theory of automorphic forms*, Princeton Univ. Press, (1971).
- [11] —: On modular correspondence for $Sp(n, \mathbb{Z})$ and their congruence relations, *Proc. N. A. S.*(1963), 824-828.
- [12] J. Tate: The non-existence of certain Galois extensions of \mathbb{Q} unramified outside 2. *Contemp. Math.* **174** (1994), 153-156.
- [13] H. Yoshida: Siegel's modular forms and the arithmetics of quadratic forms, *Invent. math.* **60** (1980), 193-248.