An alternative proof of a theorem about local newforms for GSp(4)

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The work [RS] presents a theory of local new- and oldforms for representations of $\mathrm{GSp}(4,F)$ with trivial central character for F a non-archimedean field of characteristic zero. This theory considers vectors fixed by the paramodular groups $\mathrm{K}(\mathfrak{p}^n)$ as defined in [RS]. Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. One of the main theorems of [RS] asserts that if V contains a non-zero vector fixed by some paramodular group $\mathrm{K}(\mathfrak{p}^n)$, i.e., π is paramodular, and N_π is the smallest such n, then the space $V(N_\pi)$ of $\mathrm{K}(\mathfrak{p}^{N_\pi})$ fixed vectors in V is one-dimensional. If π is paramodular, then any non-zero element $V(N_\pi)$ is called a newform. Other theorems of [RS] describe the information carried by newforms. In particular, it is proven in [RS] that if π is generic, then π is paramodular, and there exists a newform whose zeta integral is the L-factor $L(s,\pi)$. In this work we will give an alternative proof of the following theorem. See the introduction of [RS] for an extensive summary of the contents and proofs of [RS].

Theorem. ([RS]) *Let π be a supercuspidal, generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character and Whittaker model $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Assume that $V(n)$ is non-zero for some non-negative integer n, and let N_π be the smallest n such that $V(n)$ is non-zero. Then $V(N_\pi)$ is one-dimensional, and there exists W_π in $V(N_\pi)$ such that*

$$Z(s, W_\pi) = L(s,\pi) = 1.$$

In what follows we will use the definitions and notation of [RS]. In particular, let $\mathfrak{o}$ be the ring of integers of F, let $\mathfrak{p}$ be the maximal ideal of $\mathfrak{o}$, let q be the number of elements of $\mathfrak{o}/\mathfrak{p}$, fix a generator ϖ of $\mathfrak{p}$, and let ψ be a non-trivial character of F with conductor $\mathfrak{o}$.

1 A Useful Realization

Our alternative proof of the above theorem is based on an alternative realization of paramodular vectors. This realization depends on the η Principle proven in [RS]. Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character. We will work in the Whittaker model $\mathcal{W}(\pi, \psi_{c_1,c_2})$ of π. The η Principle asserts that if W is a non-zero vector in $V(n)$ for some

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non-negative integer n and W is degenerate, i.e., $Z(s,W)=0$, then $n \geq 2$ and there exists W_1 in $V(n-2)$ such that $W=\eta W_1$. Here, η is the level raising operator that increases the level by 2 and is given by the action of the group element with the same name:

$$\eta = \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}.$$

Since it is given by the action of a single group element, the level raising operator η is obviously injective. Besides vectors of the form $\eta W = \pi(\eta)W$, in what follows we will often encounter vectors of the form $\pi(\eta^{-1})W$. The reader should note that $\pi(\eta^{-1})W$ may not be paramodular even if W is paramodular. Indeed, the η Principle asserts that if W is paramodular and non-zero, then $\pi(\eta^{-1})W$ is paramodular if and only if the level of W is at least 2 and W is degenerate. To obtain another model for paramodular vectors using the η Principle, let

$$\Delta_{ij} = \begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix}$$

for integers i and j. For n a non-negative integer, W in $V(n)$ and $0 \leq i,j < \infty$ define

$$m(W)_{ij} = W(\Delta_{ij})$$

and let $m(W)$ be the matrix

$$m(W) = (m(W)_{ij})_{0\leq i,j<\infty}.$$

The connection between $m(W)$ and η is provided by the observation that

$$W(\Delta_{ij}) = (\pi(\eta^{-i})W)(\Delta_{0j}) \tag{1}$$

for all i and j with $0 \leq i,j < \infty$ and $W \in V(n)$. Thus, the i-th row of $m(W)$ is obtained by evaluating the vector $\pi(\eta^{-i})W$ at the points Δ_{0j} for $0 \leq j < \infty$. We denote by $M(n)$ the $\mathbb{C}$ vector space of all $m(W)$ for $W \in V(n)$. Using the η Principle, we can prove that $M(n)$ is a model of $V(n)$.

Proposition 1.1. *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, and let $V = \mathcal{W}(\pi,\psi_{c_1,c_2})$. For each non-negative integer n the map*

$$V(n) \xrightarrow{\sim} M(n).$$

that sends W to $m(W)$ is an isomorphism of vector spaces.

Proof. Let $W \in V(n)$ be non-zero. Thanks to the η Principle, Theorem 4.3.7 of [RS], we can write $W=\eta^i W_1$ for some non-negative integer i and $W_1 \in V(n-2i)$ with $Z(s,W_1) \neq 0$. We will prove that the i-th row of $m(W)$ is non-zero. By (1), the i-th row of $m(W)$ is

$$W_1(\Delta_{0j}), \quad 0 \leq j < \infty.$$

By Sect. 4.1 of [RS] we have

$$Z(s, W_1) = (1 - q^{-1}) \sum_{j=0}^{\infty} q^{3j/2} W_1(\Delta_{0j})(q^{-s})^j.$$

Since $Z(s, W_1) \neq 0$ we have $W_1(\Delta_{0j}) \neq 0$ for some non-negative j, so that the i-th row of $m(W)$ is non-zero. □

If π is a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$ is the Whittaker model of π, n is a non-negative integer, and $W \in V(n)$, then the matrix $m(W)$ may have infinitely many non-zero entries. However, as the next proposition shows, if π is supercuspidal, then $m(W)$ has only finitely many non-zero entries.

Proposition 1.2. *Let π be a supercuspidal, generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$, and let $n \geq 0$ be a non-negative integer. If $W \in V(n)$, then $m(W)$ has finitely many non-zero entries.*

Proof. We use the observations and notation from the proof of Proposition 2.6.4 of [RS] which involve P_3-theory. By that proof, keeping in mind that $V_2 = V_{Z^J}$ because π is supercuspidal, there exists a surjective linear map

$$V \to \mathrm{c} - \mathrm{Ind}_{U_3}^{P_3} \Theta$$

such that if W maps to f, then $W(q) = f(i(q))$ for q in the Klingen parabolic subgroup Q of $\mathrm{GSp}(4, F)$ and $i : Q \to P_3$ the surjective homomorphism from Lemma 2.5.1 of [RS]. Let $W \in V$ and let W map to f. Then

$$W(\Delta_{ij}) = f(\begin{bmatrix} \varpi^{i+j} & & \\ & \varpi^i & \\ & & 1 \end{bmatrix})$$

for any integers i and j. Since f is left invariant under a compact, open subgroup of P_3 and is compactly supported modulo the subgroup U_3, the above quantity is non-zero for only finitely many i and j. □

In the remainder of this section we translate some of the operators that act on paramodular vectors to the new model $M(n)$. These operators include the level raising operators η, θ and θ'. However, we will also need to describe a formula involving a certain level lowering operator in terms of the new model.

To give the formulas we need some notation. Let $M_{\infty\times\infty}(\mathbb{C})$ be the set of all matrices $(m_{ij})_{0\leq i,j<\infty}$ with $m_{ij} \in \mathbb{C}$. The space $M(n)$ is contained in $M_{\infty\times\infty}(\mathbb{C})$. It will be convenient to write the elements A of $M_{\infty\times\infty}(\mathbb{C})$ as a column of rows,

$$A = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}.$$

We define two shift operations Left and Right on row vectors,

$$\mathrm{Left}(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots),$$

$$\mathrm{Right}(a_0, a_1, a_2, \dots) = (0, a_0, a_1, \dots).$$

Using this notation we can describe the level raising operators θ, θ' and η in the alternative model.

Proposition 1.3. *Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. For each non-negative integer n define*

$$\theta, \theta', \eta : M_{\infty\times\infty}(\mathbb{C}) \to M_{\infty\times\infty}(\mathbb{C})$$

by

$$\theta(\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}) = q \begin{bmatrix} 0 \\ \mathrm{Left}(r_0) \\ \mathrm{Left}(r_1) \\ \vdots \end{bmatrix} + \begin{bmatrix} \mathrm{Right}(r_0) \\ \mathrm{Right}(r_1) \\ \mathrm{Right}(r_2) \\ \vdots \end{bmatrix}, \quad \theta'(\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}) = q \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix} + \begin{bmatrix} 0 \\ r_0 \\ r_1 \\ \vdots \end{bmatrix}.$$

and

$$\eta(\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}) = \begin{bmatrix} 0 \\ r_0 \\ r_1 \\ \vdots \end{bmatrix}.$$

The diagrams

$$\begin{array}{ccc} V(n) & \xrightarrow{\sim} & M(n) \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta} \\ V(n+1) & \xrightarrow{\sim} & M(n+1) \end{array}, \quad \begin{array}{ccc} V(n) & \xrightarrow{\sim} & M(n) \\ {\scriptstyle\theta'}\downarrow & & \downarrow{\scriptstyle\theta'} \\ V(n+1) & \xrightarrow{\sim} & M(n+1) \end{array}$$

and

$$\begin{array}{ccc} V(n) & \xrightarrow{\sim} & M(n) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta} \\ V(n+2) & \xrightarrow{\sim} & M(n+2) \end{array}$$

commute

Proof. This follows by direct computations using the explicit formulas from Sect. 3.2 of [RS]. □

The work [RS] also introduced a certain level lowering operator δ_1 that reduces the level by 1, and we will need a formula involving δ_1 in the setting of the alternative model. Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. Let n be an integer with $n \geq 1$. Then $\delta_1 : V(n) \to V(n-1)$ is the natural trace operator, defined by the formula

$$\delta_1 v = \sum_{g \in \mathrm{K}(\mathfrak{p}^{n-1})/(\mathrm{K}(\mathfrak{p}^{n-1}) \cap \mathrm{K}(\mathfrak{p}^n))} \pi(g)v.$$

We first present and prove the relevant formula in an abstract form.

Proposition 1.4. *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let n be an integer with $n \geq 2$. If $v \in V(n)$, then*

$$\begin{aligned}
\eta\delta_1 v &= \delta_1\theta' v - q^2 v - q^3 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & \lambda & \mu & \kappa\varpi^{-n} \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \eta^{-1})\theta' v \, d\lambda \, d\mu \, d\kappa \\
&\quad + q^2 \int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix}) v \, d\lambda \, d\mu.
\end{aligned}$$

Proof. We have by (3.3.7) of [RS]

$$\begin{aligned}
\eta\delta_1 v &= q^3\pi(\eta) \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ \mu\varpi^{n-1}\mu & & 1 & \\ \kappa\varpi^{n-1} & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix}) v \, d\lambda \, d\mu \, d\kappa \\
&\quad + q^2\pi(\eta) \int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & \lambda & \mu & \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \eta^{-1}) v \, d\lambda d\mu.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\eta\delta_1 v &= q^3 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n} & 1 & & \\ \mu\varpi^{n} & & 1 & \\ \kappa\varpi^{n+1} & \mu\varpi^{n} & -\lambda\varpi^{n} & 1 \end{bmatrix})\pi(\eta) v \, d\lambda \, d\mu \, d\kappa \\
&\quad + q^2 \int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix}) v \, d\lambda \, d\mu \\
&= q^3 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{o}^\times} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n} & 1 & & \\ \mu\varpi^{n} & & 1 & \\ \kappa\varpi^{n+1} & \mu\varpi^{n} & -\lambda\varpi^{n} & 1 \end{bmatrix})\pi(\eta) v \, d\lambda \, d\mu \, d\kappa \\
&\quad + q^3 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{p}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n} & 1 & & \\ \mu\varpi^{n} & & 1 & \\ \kappa\varpi^{n+1} & \mu\varpi^{n} & -\lambda\varpi^{n} & 1 \end{bmatrix})\pi(\eta) v \, d\lambda \, d\mu \, d\kappa \\
&\quad + q^2 \int_{\mathfrak{o}}\int_{\mathfrak{o}} \begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix}) v \, d\lambda \, d\mu
\end{aligned}$$

Applying the identity (2.8) from [RS] we have:

$$
\begin{aligned}
\eta\delta_1 v = q^3 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{o}^\times} \pi(&\begin{bmatrix} 1 & & & \\ \lambda\varpi^n & 1 & & \\ \mu\varpi^n & & 1 & \\ & \mu\varpi^n & -\lambda\varpi^n & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \kappa^{-1}\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
&\times \begin{bmatrix} -\kappa^{-1}\varpi^{-(n+1)} & & & \\ & 1 & & \\ & & 1 & \\ & & & -\kappa\varpi^{n+1} \end{bmatrix} \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{bmatrix} \\
&\times \begin{bmatrix} 1 & & & \kappa^{-1}\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\pi(\eta)v \, d\lambda \, d\mu \, d\kappa \\
+ q^2 \int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(&\begin{bmatrix} 1 & & & \\ \lambda\varpi^n & 1 & & \\ \mu\varpi^n & & 1 & \\ & \mu\varpi^n & -\lambda\varpi^n & 1 \end{bmatrix})\pi(\eta)v \, d\lambda \, d\mu \\
+ q^2 \int_{\mathfrak{o}}\int_{\mathfrak{o}} &\begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix})v \, d\lambda \, d\mu \\
= q^3 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{o}^\times} \pi(&\begin{bmatrix} 1 & & & \\ \lambda\varpi^n & 1 & & \\ \mu\varpi^n & & 1 & \\ & \mu\varpi^n & -\lambda\varpi^n & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \kappa\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v \, d\lambda \, d\mu \, d\kappa \\
+ q^2 \int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(&\begin{bmatrix} 1 & & & \\ \lambda\varpi^n & 1 & & \\ \mu\varpi^n & & 1 & \\ & \mu\varpi^n & -\lambda\varpi^n & 1 \end{bmatrix})\pi(\eta)v \, d\lambda \, d\mu \\
+ q^2 \int_{\mathfrak{o}}\int_{\mathfrak{o}} &\begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix})v \, d\lambda \, d\mu \\
= q^3 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(&\begin{bmatrix} 1 & & & \\ \lambda\varpi^n & 1 & & \\ \mu\varpi^n & & 1 & \\ & \mu\varpi^n & -\lambda\varpi^n & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \kappa\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v \, d\lambda \, d\mu \, d\kappa \\
- q^3 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{p}} \pi(&\begin{bmatrix} 1 & & & \\ \lambda\varpi^n & 1 & & \\ \mu\varpi^n & & 1 & \\ & \mu\varpi^n & -\lambda\varpi^n & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \kappa\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v \, d\lambda \, d\mu \, d\kappa \\
+ q^2 \int_{\mathfrak{o}}\int_{\mathfrak{o}} \pi(&\begin{bmatrix} 1 & & & \\ \lambda\varpi^n & 1 & & \\ \mu\varpi^n & & 1 & \\ & \mu\varpi^n & -\lambda\varpi^n & 1 \end{bmatrix})\pi(\eta)v \, d\lambda \, d\mu \, d\kappa
\end{aligned}
$$

$$
+ q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix}) v \, d\lambda \, d\mu
$$

$$
= q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n} & 1 & & \\ \mu\varpi^{n} & & 1 & \\ & \mu\varpi^{n} & -\lambda\varpi^{n} & 1 \end{bmatrix}) (\sum_{x \in \mathfrak{o}} \begin{bmatrix} 1 & & & x\varpi^{-(n+1)} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) v) \, d\lambda \, d\mu
$$

$$
- q^2 v
$$

$$
+ q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n} & 1 & & \\ \mu\varpi^{n} & & 1 & \\ & \mu\varpi^{n} & -\lambda\varpi^{n} & 1 \end{bmatrix}) \pi(\eta) v \, d\lambda \, d\mu \, d\kappa
$$

$$
+ q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix}) v \, d\lambda \, d\mu
$$

$$
= q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n} & 1 & & \\ \mu\varpi^{n} & & 1 & \\ & \mu\varpi^{n} & -\lambda\varpi^{n} & 1 \end{bmatrix}) (\theta' v - \eta v) \, d\lambda \, d\mu
$$

$$
- q^2 v
$$

$$
+ q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n} & 1 & & \\ \mu\varpi^{n} & & 1 & \\ & \mu\varpi^{n} & -\lambda\varpi^{n} & 1 \end{bmatrix}) \pi(\eta) v \, d\lambda \, d\mu \, d\kappa
$$

$$
+ q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix}) v \, d\lambda \, d\mu
$$

$$
= q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n} & 1 & & \\ \mu\varpi^{n} & & 1 & \\ & \mu\varpi^{n} & -\lambda\varpi^{n} & 1 \end{bmatrix}) \theta' v \, d\lambda \, d\mu
$$

$$
- q^2 v
$$

$$
+ q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix}) v \, d\lambda \, d\mu
$$

$$
= \delta_1 \theta' v - q^2 v - q^3 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & \lambda & \mu & \kappa\varpi^{-n} \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \eta^{-1}) \theta' v \, d\lambda \, d\mu \, d\kappa
$$

$$+ q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} \begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix}) v \, d\lambda d\mu.$$

The last equality follows from (3.23) of [RS]. □

The next corollary translates the last proposition to the setting of the alternative model for $V(n)$. In contrast to the previous proposition, the alternative model requires that the representation is generic.

Corollary 1.5. *Let (π, V) be a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Define*

$$J : M_{\infty\times\infty}(\mathbb{C}) \to M_{\infty\times\infty}(\mathbb{C})$$

by

$$J(A) = \begin{bmatrix} r_0 + q^2 r_1 \\ q^2 r_2 \\ q^2 r_3 \\ \vdots \end{bmatrix} \quad \textit{for} \quad A = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}.$$

Let n be a non-negative integer with $n \geq 2$. We have for $W \in V(n)$,

$$m(\eta\delta_1 W) = m(\delta_1\theta' W) - q^3 m(W) - q^2 J(m(W)).$$

If $A \in m(\ker \delta_1)$, then $J(A) \in M(n)$. The diagram

$$\begin{array}{ccc} \ker(\delta_1) & \xrightarrow{\sim} & m(\ker \delta_1) \\ {\scriptstyle q^{-2}\delta_1\theta' - q\cdot \mathrm{Id}} \downarrow & & \downarrow {\scriptstyle J} \\ V(n) & \xrightarrow{\sim} & M(n). \end{array}$$

commutes.

Proof. We apply the m operator to the formula

$$\eta\delta_1 W = \delta_1\theta' W - q^2 W - q^3 \int\limits_{\mathfrak{o}} \int\limits_{\mathfrak{o}} \int\limits_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & \lambda & \mu & \kappa\varpi^{-n} \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \eta^{-1})\theta' W \, d\lambda \, d\mu \, d\kappa$$

$$+ q^2 \int\limits_{\mathfrak{o}} \int\limits_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix}) W \, d\lambda \, d\mu$$

from Proposition 1.4 by evaluating both sides of this formula at the element Δ_{ij} for $0 \leq i, j < \infty$. We have

$$- q^3 \int\limits_{\mathfrak{o}} \int\limits_{\mathfrak{o}} \int\limits_{\mathfrak{o}} (\theta' W)(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda & \mu & \kappa\varpi^{-n} \\ & 1 & & \mu \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \eta^{-1}) \, d\lambda \, d\mu \, d\kappa$$

$$= -q^3 \int_{\mathfrak{o}} (\theta' W)(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda & & \\ & 1 & & \\ & & 1 & -\lambda \\ & & & 1 \end{bmatrix} \eta^{-1})\, d\lambda$$

$$= -q^3 \int_{\mathfrak{o}} \psi(c_1 \lambda \varpi^i)(\theta' W)(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \eta^{-1})\, d\lambda$$

$$= -q^3 (\theta' W)(\begin{bmatrix} \varpi^{2i+j+1} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & \varpi^{-1} \end{bmatrix})$$

$$= -q^3 (\theta' W)(\begin{bmatrix} \varpi^{2i+j+2} & & & \\ & \varpi^{i+j+1} & & \\ & & \varpi^{i+1} & \\ & & & 1 \end{bmatrix}).$$

By Lemma 3.2.2 of [RS], this equals

$$- q^3 W(\begin{bmatrix} \varpi^{2i+j+1} & & & \\ & \varpi^{i+j+1} & & \\ & & \varpi^{i+1} & \\ & & & \varpi \end{bmatrix})$$

$$- q^4 W(\begin{bmatrix} \varpi^{2i+j+2} & & & \\ & \varpi^{i+j+1} & & \\ & & \varpi^{i+1} & \\ & & & 1 \end{bmatrix})$$

$$= -q^3 W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix}) - q^4 W(\begin{bmatrix} \varpi^{2i+j+2} & & & \\ & \varpi^{i+j+1} & & \\ & & \varpi^{i+1} & \\ & & & 1 \end{bmatrix})$$

$$= -q^3 m(W)_{ij} - q^4 m(W)_{i+1,j}.$$

Also,

$$q^2 \int_{\mathfrak{o}} \int_{\mathfrak{o}} W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda\varpi^{-1} & \mu\varpi^{-1} & \\ & 1 & & \mu\varpi^{-1} \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix})\, d\lambda\, d\mu$$

$$= q^2 \int_{\mathfrak{o}} W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & \lambda\varpi^{-1} & & \\ & 1 & & \\ & & 1 & -\lambda\varpi^{-1} \\ & & & 1 \end{bmatrix})\, d\lambda$$

$$= q^2 \int_{\mathfrak{o}} \psi(c_1 \varpi^{i-1} \lambda) W(\begin{bmatrix} \varpi^{2i+j} & & & \\ & \varpi^{i+j} & & \\ & & \varpi^{i} & \\ & & & 1 \end{bmatrix})\, d\lambda$$

$$= \begin{cases} 0 & \text{if } i = 0, \\ q^2 m(W)_{ij} & \text{if } i > 0. \end{cases}$$

The claims of the lemma follow from these computations. $\square$

The main application of the previous corollary will be at the minimal level N_π. At the minimal level, because the kernel of δ_1 must be all of $V(N_\pi)$, the map J is actually an endomorphism of $V(N_\pi)$.

Corollary 1.6. *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Then the endomorphism*

$$J : V(N_\pi) \to V(N_\pi)$$

is given by

$$J(A) = \begin{bmatrix} r_0 + q^2 r_1 \\ q^2 r_2 \\ q^2 r_3 \\ \vdots \end{bmatrix} \quad \textit{for} \quad A = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}$$

is an endomorphism of $V(N_\pi)$.

2 Analysis of the Second Row

In this section we expose some properties of the second row of the matrix $m(W)$ associated to a paramodular vector in a generic representation. We will use these properties, in combination with the results involving the level lowering operator δ_1 from the previous section, to give the alternative proof of the theorem from the introduction.

To analyze the second row of $m(W)$ is it useful to use zeta integrals. Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let W be a paramodular vector in V. As explained in the previous section, the second row of $m(W)$ is

$$m(W)_{1j} = W(\Delta_{1j}) = (\pi(\eta^{-1})W)(\Delta_{0j}), \quad 0 \le j < \infty.$$

The next proposition shows that these numbers are encapsulated in a certain auxiliary zeta integral.

Proposition 2.1. *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. For W in V define*

$$Z_N(s, W) = \int_{F^\times} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a.$$

If n is a non-negative integer and $W \in V(n)$, then

$$Z_N(s, \pi(\eta^{-1})W) = (1 - q^{-1}) \sum_{j=0}^{\infty} q^{3j/2} m(W)_{1j} (q^{-s})^j.$$

Proof. Let $W \in V(n)$. We claim that

$$W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \eta^{-1}) = 0$$

for $v(a) < 0$. To see this, let $a \in F^\times$ and $y \in \mathfrak{o}$. Then

$$\begin{aligned} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \eta^{-1}) &= W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \eta^{-1} \begin{bmatrix} 1 & & & \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix}) \\ &= \psi(c_2 a y) W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \eta^{-1}). \end{aligned}$$

Since ψ is non-trivial on $\mathfrak{p}^{-1}$ our claim follows. The remainder of the proposition follows by a computation. □

Given this proposition, our next goal will be to analyze the auxiliary zeta integral $Z_N(s, \pi(\eta^{-1})W)$ for a paramodular vector W. We will show that this zeta integral satisfies a certain functional equation. This will be the basis for further analysis of the second row of $m(W)$. We begin by relating $Z_N(s, \pi(\eta^{-1})W)$ to the full zeta integral $Z(s, \pi(\eta^{-1})W)$: recall that the standard zeta integral also involves an integration over F.

Lemma 2.2. *Let π be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let n be a non-negative integer and $W \in V(n)$. Then*

$$Z(s, \pi(\eta^{-1})W) = Z_N(s, \pi(\eta^{-1})W) + (q-1)q^{-3}(q^{-s})^{-2} \cdot \Big(Z(s, W) - W(1)\Big).$$

Proof. We compute:

$$\begin{aligned} &Z(s, \eta^{-1}W) \\ &= \int\limits_{F^\times} \int\limits_{F} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} \eta^{-1})|a|^{s-3/2}\, dx\, d^\times a \\ &= \int\limits_{F^\times} \int\limits_{v(x)\geq 0} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} \eta^{-1})|a|^{s-3/2}\, dx\, d^\times a \\ &\quad + \int\limits_{F^\times} \int\limits_{v(x)<0} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} \eta^{-1})|a|^{s-3/2}\, dx\, d^\times a \end{aligned}$$

$$
\begin{aligned}
&= \int_{F^\times} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \eta^{-1})|a|^{s-3/2}\, d^\times a \\
&\quad + \int_{F^\times} \int_{v(x)<0} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & x^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
&\qquad\qquad \times \begin{bmatrix} 1 & & & \\ & -x^{-1} & & \\ & & -x & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix} \\
&\qquad\qquad\quad \times \begin{bmatrix} 1 & & & \\ & 1 & x^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \eta^{-1})|a|^{s-3/2}\, dx\, d^\times a \\
&= Z_N(s, \eta^{-1} W) \\
&+ \int_{F^\times} \int_{v(x)<0} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & x^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
&\qquad\qquad \times \begin{bmatrix} 1 & & & \\ & -x^{-1} & & \\ & & -x & \\ & & & 1 \end{bmatrix} \eta^{-1})|a|^{s-3/2}\, dx\, d^\times a \\
&= Z_N(s, \eta^{-1} W) \\
&+ \int_{F^\times} \int_{v(x)<0} W(\begin{bmatrix} 1 & & & \\ & 1 & ax^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \\
&\qquad\qquad \times \begin{bmatrix} 1 & & & \\ & x^{-1} & & \\ & & x & \\ & & & 1 \end{bmatrix} \eta^{-1})|a|^{s-3/2}\, dx\, d^\times a \\
&= Z_N(s, \eta^{-1} W) \\
&+ \int_{F^\times} \int_{v(x)<0} \psi(c_2 a x^{-1}) W(\begin{bmatrix} a & & & \\ & ax^{-1} & & \\ & & x & \\ & & & 1 \end{bmatrix} \eta^{-1})|a|^{s-3/2}\, dx\, d^\times a \\
&= Z_N(s, \eta^{-1} W) \\
&+ \int_{F^\times} \int_{v(x)<0} \psi(c_2 a x^{-1}) W(\begin{bmatrix} a\varpi & & & \\ & ax^{-1} & & \\ & & x & \\ & & & \varpi^{-1} \end{bmatrix})|a|^{s-3/2}\, dx\, d^\times a.
\end{aligned}
$$

Now $v(a\varpi) < v(ax^{-1}) \iff v(x) < -1$. Hence, by Lemma 4.1.2 of [RS],

$$\begin{aligned}
Z(s,\eta^{-1}W) &= Z_N(s,\eta^{-1}W) \\
&\quad + \int_{F^\times}\int_{v(x)=-1} \psi(c_2ax^{-1})W(\begin{bmatrix} a\varpi &&& \\ & ax^{-1} && \\ && x & \\ &&& \varpi^{-1}\end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a \\
&= Z_N(s,\eta^{-1}W) \\
&\quad + \int_{F^\times}\int_{v(x)=-1} \psi(c_2ax^{-1})W(\begin{bmatrix} a\varpi &&& \\ & a\varpi && \\ && \varpi^{-1} & \\ &&& \varpi^{-1}\end{bmatrix})|a|^{s-3/2}\,dx\,d^\times a \\
&= Z_N(s,\eta^{-1}W) \\
&\quad + \int_{F^\times}(\int_{v(x)=-1} \psi(c_2ax^{-1})\,dx)W(\begin{bmatrix} a\varpi^2 &&& \\ & a\varpi^2 && \\ && 1 & \\ &&& 1\end{bmatrix})|a|^{s-3/2}\,d^\times a.
\end{aligned}$$

It is easily computed that

$$\int_{v(x)=-1} \psi(c_2ax^{-1})\,dx = \begin{cases} 0 & \text{if } v(a) < -2, \\ -1 & \text{if } v(a) = -2, \\ q-1 & \text{if } v(a) > -2. \end{cases}$$

Hence

$$\begin{aligned}
Z(s,\eta^{-1}W) &= Z_N(s,\eta^{-1}W) \\
&\quad + (-1)\int_{v(a)=-2} W(\begin{bmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1\end{bmatrix})|\varpi^{-2}|^{s-3/2}\,d^\times a \\
&\quad + (q-1)\int_{v(a)>-2} W(\begin{bmatrix} a\varpi^2 &&& \\ & a\varpi^2 && \\ && 1 & \\ &&& 1\end{bmatrix})|a|^{s-3/2}\,d^\times a \\
&= Z_N(s,\eta^{-1}W) \\
&\quad + (-1)W(1)|\varpi|^{3-2s}(\int_{v(a)=-2} d^\times a) \\
&\quad + (q-1)\int_{F^\times} \chi_{v(t)>-2}(a)W(\begin{bmatrix} a\varpi^2 &&& \\ & a\varpi^2 && \\ && 1 & \\ &&& 1\end{bmatrix})|a|^{s-3/2}\,d^\times a \\
&= Z_N(s,\eta^{-1}W) + (-1)W(1)(1-q^{-1})|\varpi|^{3-2s} \\
&\quad + (q-1)\int_{F^\times} \chi_{v(t)>-2}(a\varpi^{-2})W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix})|a\varpi^{-2}|^{s-3/2}\,d^\times a
\end{aligned}$$

$$
\begin{aligned}
&= Z_N(s,\eta^{-1}W) + (-1)W(1)(1-q^{-1})|\varpi|^{3-2s} \\
&\quad + (q-1)|\varpi|^{3-2s} \int\limits_{F^\times} \chi_{v(t)>0}(a) W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a \\
&= Z_N(s,\eta^{-1}W) + (-1)W(1)(1-q^{-1})|\varpi|^{3-2s} \\
&\quad + (q-1)|\varpi|^{3-2s}(Z(s,W) - \int\limits_{v(a)=0} W(1)\, d^\times a) \\
&= Z_N(s,\eta^{-1}W) + (-1)W(1)(1-q^{-1})|\varpi|^{3-2s} \\
&\quad + (q-1)|\varpi|^{3-2s}(Z(s,W) - (1-q^{-1})W(1)) \\
&= Z_N(s,\eta^{-1}W) + Z(s,W)(q-1)|\varpi|^{3-2s} \\
&\quad + (-1)W(1)(1-q^{-1})|\varpi|^{3-2s} - (q-1)(1-q^{-1})W(1)|\varpi|^{3-2s} \\
&= Z_N(s,\eta^{-1}W) + Z(s,W)(q-1)|\varpi|^{3-2s} \\
&\quad + (-(1-q^{-1}) - (q-1)(1-q^{-1}))W(1)|\varpi|^{3-2s} \\
&= Z_N(s,\eta^{-1}W) + Z(s,W)(q-1)|\varpi|^{3-2s} \\
&\quad - (q-1)W(1)|\varpi|^{3-2s} \\
&= Z_N(s,\eta^{-1}W) + (q-1)|\varpi|^{3-2s}(Z(s,W) - W(1)).
\end{aligned}
$$

This completes the proof. □

Next, we present the functional equation satified by the auxiliary zeta integral. This requires the introduction of a new concept, namely an operator on meromorphic functions on the complex plane having to do with functional equations. Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and let $V = \mathcal{W}(\pi,\psi_{c_1,c_2})$. If n is a non-negative integer, then we define the operator $u_n[\cdot]$ on the vector space of meromorphic functions on $\mathbb{C}$ by the formula

$$
u_n\big[f(s)\big] = q^{n/2}(q^{-s})^n \gamma(1-s,\pi) f(1-s).
$$

A computation shows that

$$
u_n\big[u_n\big[f(s)\big]\big] = f(s)
$$

for any meromorphic function on the complex plane. Moreover, if W is in V, then

$$
u_n\big[Z(s,W)\big] = Z(s,\pi(u_n)W).
$$

This is a translation of the functional equation for zeta integrals.

Proposition 2.3. *Let π be a generic, irreducible, admissible representation of $\mathrm{GSp}(4,F)$ with trivial central character, and let $V = \mathcal{W}(\pi,\psi_{c_1,c_2})$. Let n be a non-negative integer such that $n \geq 2$ and let $W \in V(n)$. Then*

$$
\begin{aligned}
&(q^{-s})^2 Z_N(s,\pi(\eta^{-1})W) - q^{-1}u_n\big[Z_N(s,\pi(\eta^{-1})\pi(u_n)W)\big] \\
&= (q-1)q^{-2}((q^{-s})^2 - q^{-1})Z(s,W) \\
&\qquad - (q-1)q^{-2}\big(u_n\big[(\pi(u_n)W)(1)\big](q^{-s})^2 - W(1)q^{-1}\big).
\end{aligned}
$$

Proof. The identity

$$\eta^{-1}u_n = \begin{bmatrix} \varpi & & & \\ & \varpi & & \\ & & \varpi & \\ & & & \varpi \end{bmatrix} u_{n-2}\eta^{-1}$$

implies that $\pi(\eta^{-1}u_n) = \pi(u_{n-2}\eta^{-1})$. Therefore,

$$Z(s, \pi(\eta^{-1}u_n)W) = Z(s, \pi(u_{n-2}\eta^{-1})W).$$

We will compute both sides of this equation using Lemma 2.2. First of all,

$$\begin{aligned} Z(s, \pi(\eta^{-1}u_n)W) &= Z_N(s, \pi(\eta^{-1}u_n)W) \\ &\quad + (q-1)q^{-3}(q^{-s})^{-2}(Z(s, \pi(u_n)W) - (\pi(u_n)W)(1)). \end{aligned}$$

And using Lemma 2.2,

$$\begin{aligned}
& Z(s, \pi(u_{n-2}\eta^{-1})W) \\
&= Z(s, \pi(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \varpi^{n-2} & \\ & & & \varpi^{n-2} \end{bmatrix} u_0\eta^{-1})W) \\
&= Z(s, \pi(\begin{bmatrix} \varpi^{-(n-2)} & & & \\ & \varpi^{-(n-2)} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} u_0\eta^{-1})W) \\
&= |\varpi^{-(n-2)}|^{1/2-s} Z(s, \pi(u_0\eta^{-1})W) \\
&= |\varpi|^{(n-2)(s-1/2)} Z(s, \pi(u_0\eta^{-1})W) \\
&= |\varpi|^{(n-2)(s-1/2)} \gamma(1-s) Z(1-s, \pi(\eta^{-1})W) \\
&= |\varpi|^{(n-2)(s-1/2)} \gamma(1-s) \Big(Z_N(1-s, \pi(\eta^{-1})W) \\
&\qquad + (q-1)q^{-3}(q^{-(1-s)})^{-2}(Z(1-s, W) - W(1)) \Big) \\
&= (q^{-s})^{-2} q^{-1} q^{-ns} q^{n/2} \gamma(1-s) \Big(Z_N(1-s, \pi(\eta^{-1})W) \\
&\qquad + (q-1)q^{-1}q^{-2s}(Z(1-s, W) - W(1)) \Big) \\
&= (q^{-s})^{-2} q^{-1} q^{-ns} q^{n/2} \gamma(1-s) Z_N(1-s, \pi(\eta^{-1})W) \\
&\qquad + (q-1)q^{-2}q^{-ns}q^{n/2}\gamma(1-s) Z(1-s, W) \\
&\qquad - (q-1)q^{-2}q^{-ns}q^{n/2}\gamma(1-s) W(1) \\
&= (q^{-s})^{-2} q^{-1} u_n \big[Z_N(s, \pi(\eta^{-1})W) \big] \\
&\qquad + (q-1)q^{-2} u_n \big[Z(s, W) \big] \\
&\qquad - (q-1)q^{-2} u_n \big[W(1) \big] \\
&= (q^{-s})^{-2} q^{-1} u_n \big[Z_N(s, \pi(\eta^{-1})W) \big] \\
&\qquad + (q-1)q^{-2}(Z(s, \pi(u_n)W) - u_n\big[W(1)\big]).
\end{aligned}$$

Equating and multiplying by $(q^{-s})^2$ now produces an equation. If $\pi(u_n)W$ is substituted in this equation for W then the result follows. □

More work is required to exploit the functional equation involving the auxiliary zeta integral $Z_N(s, \pi(\eta^{-1})W)$. Our next goal will be to prove that the factor

$$u_n\big[Z_N(s, \pi(\eta^{-1})\pi(u_n)W)\big]$$

from the functional equation is actually $Z_N(s, \pi(\eta^{-1})W)$ under the assumption that $\delta_1 W = 0$ and $\delta_1 \pi(u_n)W = 0$. Here, δ_1 is the level lowering operator mentioned in the previous section. This will make for a simpler functional equation, and will be applicable at the minimal paramodular level N_π; we will also apply it to some vectors at level $N_\pi + 1$. In what follows we use a certain operator R introduced in Sect. 7.3 of [RS]. Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let W be in V. Then we set

$$RW = q\int_{\mathfrak{o}} \pi(\begin{bmatrix} 1 &&& \\ \lambda\varpi^{n-1} & 1 && \\ && 1 & \\ && -\lambda\varpi^{n-1} & 1 \end{bmatrix})W\,d\lambda.$$

As always, we use the Haar measure on F that assigns $\mathfrak{o}$ measure one. The next lemma relates the auxiliary zeta integral to the zeta integral of $\delta_1 W$ and RW. This lemma will be the basis for proving that the above factor is $Z_N(s, \pi(\eta^{-1})W)$ under the mentioned conditions, though more work about zeta integrals involving RW will also be required.

Lemma 2.4. *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let n be a non-nonegative integer with $n \geq 2$, and let $W \in V(n)$. then*

$$Z(s, \delta_1 W) = q^3 Z_N(s, \pi(\eta^{-1})W) + Z_N(s, RW),$$

Proof. Recall from Lemma 3.3.7 of [RS] that $\delta_1 W = W_1 + W_2$ with

$$W_1 = q^3 \int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}} \pi(\begin{bmatrix} 1 & \lambda & \mu & \kappa\varpi^{1-n} \\ & 1 && \mu \\ && 1 & -\lambda \\ &&& 1 \end{bmatrix}\eta^{-1})d\lambda\,d\mu\,d\kappa,$$

$$W_2 = q^2 \int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}} \pi(\begin{bmatrix} 1 &&& \\ \lambda\varpi^{n-1} & 1 && \\ \mu\varpi^{n-1} && 1 & \\ & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix})d\lambda\,d\mu.$$

By Lemma 4.1.1 of [RS],

$$Z(s, \delta_1 W) = Z_N(s, \delta_1 W) = Z_N(s, W_1) + Z_N(s, W_2).$$

By the Whittaker transformation property,

$$Z_N(s, W_1) = \int\limits_{F^\times} W_1(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\,d^\times a$$

$$= q^3 \int_{F^\times}\int_{\mathfrak{o}} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix} 1 & \lambda && \\ & 1 && \\ && 1 & -\lambda \\ &&& 1\end{bmatrix}\eta^{-1})|a|^{s-3/2}\,d\lambda\,d^\times a$$

$$= q^3 \int_{F^\times}\int_{\mathfrak{o}} \psi(c_1\lambda)\,W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}\eta^{-1})|a|^{s-3/2}\,d\lambda\,d^\times a$$

$$= q^3 \int_{F^\times} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}\eta^{-1})|a|^{s-3/2}\,d^\times a$$

$$= q^3\, Z_N(s, \pi(\eta^{-1})W).$$

This is the first term on the right side of the asserted equality. The matrix identity

$$\begin{bmatrix} 1 &&& \\ \lambda\varpi^{n-1} & 1 && \\ \mu\varpi^{n-1} && 1 & \\ & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1\end{bmatrix} = \begin{bmatrix} 1 & -x\mu\varpi^{-1} & x\lambda\varpi^{-1} & x\varpi^{-n} \\ & 1 && x\lambda\varpi^{-1} \\ && 1 & x\mu\varpi^{-1} \\ &&& 1\end{bmatrix}$$
$$\times \begin{bmatrix} 1 &&& \\ \lambda\varpi^{n-1} & 1 && \\ \mu\varpi^{n-1} && 1 & \\ & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1\end{bmatrix}$$
$$\times \begin{bmatrix} 1 &&& -x\varpi^{-n} \\ & 1 - x\lambda\mu\varpi^{n-2} & x\lambda^2\varpi^{n-2} & \\ & -x\mu^2\varpi^{n-2} & 1 + x\lambda\mu\varpi^{n-2} & \\ &&& 1\end{bmatrix}$$

shows that

$$W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix} 1 &&& \\ \lambda\varpi^{n-1} & 1 && \\ \mu\varpi^{n-1} && 1 & \\ & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1\end{bmatrix})$$
$$= \psi(-c_1x\mu\varpi^{-1})W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix} 1 &&& \\ \lambda\varpi^{n-1} & 1 && \\ \mu\varpi^{n-1} && 1 & \\ & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1\end{bmatrix})$$

for all $x \in \mathfrak{o}$. Therefore, if μ is a unit, the above is zero. Hence

$$Z_N(s, W_2) = \int_{F^\times} W_2(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix})|a|^{s-3/2}\,d^\times a$$

$$= q^2 \int_{F^\times}\int_{\mathfrak{p}}\int_{\mathfrak{o}} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ \mu\varpi^{n-1} & & 1 & \\ & \mu\varpi^{n-1} & -\lambda\varpi^{n-1} & 1 \end{bmatrix})|a|^{s-3/2}\,d\lambda\,d\mu\,d^\times a$$

$$= q\int\limits_{F^\times}\int\limits_{\mathfrak{o}} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ \lambda\varpi^{n-1} & 1 & & \\ & & 1 & \\ & & -\lambda\varpi^{n-1} & 1 \end{bmatrix})|a|^{s-3/2}\,d\lambda\,d^\times a$$

$$= Z_N(s, RW).$$

This proves the lemma. □

Next, we relate $Z_N(s, RW)$ to $Z(s, RW)$.

Lemma 2.5. *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let n be a non-nonegative integer with $n \geq 2$, and let $W \in V(n)$. Then*

$$Z(s, RW) = q^{-1}Z_N(s, RW) + (1 - q^{-1})Z(s, W).$$

Proof. We have

$Z(s, RW)$

$$= q\int\limits_{F^\times}\int\limits_{F}\int\limits_{\mathfrak{o}} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ \mu\varpi^{n-1} & 1 & & \\ & & 1 & \\ & & -\mu\varpi^{n-1} & 1 \end{bmatrix})|a|^{s-3/2}\,d\mu\,dx\,d^\times a.$$

Let

$$A = q\int\limits_{F^\times}\int\limits_{v(x)\geq 1}\int\limits_{\mathfrak{o}} \dots\, d\mu\,dx\,d^\times a, \qquad B = q\int\limits_{F^\times}\int\limits_{v(x)<1}\int\limits_{\mathfrak{o}} \dots\, d\mu\,dx\,d^\times a.$$

We compute

$$A = q\int\limits_{F^\times}\int\limits_{v(x)\geq 1}\int\limits_{\mathfrak{o}} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ \mu\varpi^{n-1} & 1 & & \\ & & 1 & \\ & & -\mu\varpi^{n-1} & 1 \end{bmatrix})|a|^{s-3/2}\,d\mu\,dx\,d^\times a$$

$$= q\int\limits_{F^\times}\int\limits_{v(x)\geq 1}\int\limits_{\mathfrak{o}} W(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & & & \\ \mu\varpi^{n-1} & 1 & & \\ & & 1 & \\ & & -\mu\varpi^{n-1} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & & & \\ & 1 & & \\ \varpi^{n-1}x\mu & x & 1 & \\ \varpi^{2n-2}x\mu^2 & \varpi^{n-1}x\mu & & 1 \end{bmatrix})|a|^{s-3/2}\,d\mu\,dx\,d^\times a$$

$$= q\int_{F^\times}\int_{v(x)\geq 1}\int_{\mathfrak{o}} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1\end{bmatrix})|a|^{s-3/2}\,d\mu\,dx\,d^\times a$$

$$= q\cdot q^{-1}\int_{F^\times}\int_{\mathfrak{o}} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1\end{bmatrix})|a|^{s-3/2}\,d\mu\,d^\times a$$

$$= q^{-1}Z_N(s,RW).$$

This is the first term on the right side of the asserted equality. Next we compute

$$B = q\int_{F^\times}\int_{v(x)<1}\int_{\mathfrak{o}} W(\begin{bmatrix} a &&& \\ & a && \\ & x & 1 & \\ &&& 1\end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1\end{bmatrix})|a|^{s-3/2}\,d\mu\,dx\,d^\times a$$

$$= q\int_{F^\times}\int_{v(x)<1}\int_{\mathfrak{o}} W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 & x^{-1} & \\ && 1 & \\ &&& 1\end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ & -x^{-1} && \\ && -x & \\ &&& 1\end{bmatrix}\begin{bmatrix} 1 &&& \\ && 1 & \\ & -1 && \\ &&& 1\end{bmatrix}\begin{bmatrix} 1 &&& \\ & 1 & x^{-1} & \\ && 1 & \\ &&& 1\end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1\end{bmatrix})|a|^{s-3/2}\,d\mu\,dx\,d^\times a$$

$$= q\int_{F^\times}\int_{v(x)<1}\int_{\mathfrak{o}} W(\begin{bmatrix} 1 &&& \\ & 1 & ax^{-1} & \\ && 1 & \\ &&& 1\end{bmatrix}\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1\end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ & -x^{-1} && \\ && -x & \\ &&& 1\end{bmatrix} s_2 \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1\end{bmatrix})|a|^{s-3/2}\,d\mu\,dx\,d^\times a$$

$$= q \int_{F^\times} \int_{v(x)<1} \int_{\mathfrak{o}} \psi(c_2 a x^{-1}) W(\begin{bmatrix} a &&& \\ & -ax^{-1} && \\ && -x & \\ &&& 1 \end{bmatrix} s_2$$

$$\times \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1 \end{bmatrix})|a|^{s-3/2} \, d\mu \, dx \, d^\times a$$

$$= q \int_{F^\times} \int_{v(x)<1} \int_{\mathfrak{o}} \psi(c_2 a x^{-1}) W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \varpi^{-v(x)} && \\ && \varpi^{v(x)} & \\ &&& 1 \end{bmatrix} s_2$$

$$\times \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1 \end{bmatrix})|a|^{s-3/2} \, d\mu \, dx \, d^\times a.$$

Let $y \in \varpi^{-1}\mathfrak{o}$, $a \in F^\times$, $v(x) < 1$ and $\mu \in \mathfrak{o}$. Then

$$\psi(c_2 y) W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \varpi^{-v(x)} && \\ && \varpi^{v(x)} & \\ &&& 1 \end{bmatrix} s_2$$

$$\times \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1 \end{bmatrix})$$

$$= W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \varpi^{-v(x)} && \\ && \varpi^{v(x)} & \\ &&& 1 \end{bmatrix} s_2 \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 &&& \\ & 1 && \\ -a^{-1}\varpi^{n-1+2v(x)} y\mu & -a^{-1}\varpi^{2v(x)} y & 1 & \\ -a^{-1}\varpi^{2n-2+2v(x)} y\mu^2 & -a^{-1}\varpi^{n-1+2v(x)} y\mu && 1 \end{bmatrix})$$

If $2v(x) \geq v(a) + 2$, then the rightmost matrix is in $\mathrm{K}(\mathfrak{p}^n)$, implying that the above is zero. Similarly,

$$\psi(c_1 y) W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \varpi^{-v(x)} && \\ && \varpi^{v(x)} & \\ &&& 1 \end{bmatrix} s_2$$

$$\times \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1 \end{bmatrix})$$

$$= W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \varpi^{-v(x)} && \\ && \varpi^{v(x)} & \\ &&& 1 \end{bmatrix} s_2 \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 && y\varpi^{-v(x)} & \\ & 1 & -2\varpi^{n-1-v(x)}\mu y & y\varpi^{-v(x)} \\ && 1 & \\ &&& 1 \end{bmatrix}).$$

If $-1 \geq v(x)$ then the rightmost matrix is in $\mathrm{K}(\mathfrak{p}^n)$, implying that the above is zero. Therefore,

$$B = q \int\limits_{F^\times} \int\limits_{\substack{v(x)<1 \\ 2v(x)<v(a)+2 \\ -1<v(x)}} \int\limits_{\mathfrak{o}} \psi(c_2 a x^{-1}) W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} \begin{bmatrix} 1 &&& \\ & \varpi^{-v(x)} && \\ && \varpi^{v(x)} & \\ &&& 1 \end{bmatrix}$$

$$\times s_2 \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1 \end{bmatrix})|a|^{s-3/2}\, d\mu\, dx\, d^\times a$$

$$= q \int\limits_{-2<v(a)} \int\limits_{v(x)=0} \int\limits_{\mathfrak{o}} \psi(c_2 a x^{-1}) W(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix}$$

$$\times s_2 \begin{bmatrix} 1 &&& \\ \mu\varpi^{n-1} & 1 && \\ && 1 & \\ && -\mu\varpi^{n-1} & 1 \end{bmatrix})|a|^{s-3/2}\, d\mu\, dx\, d^\times a$$

$$= \int\limits_{-2<v(a)} \big(\int\limits_{\mathfrak{o}^\times} \psi(c_2 a x^{-1})\, dx \big) (\pi(s_2) R W)(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a.$$

Now

$$\int\limits_{\mathfrak{o}^\times} \psi(c_2 a x^{-1})\, dx = \begin{cases} 0 & \text{if } v(a) < 1, \\ -q^{-1} & \text{if } v(a) = -1, \\ 1 - q^{-1} & \text{if } v(a) > -1. \end{cases}$$

Hence,

$$B = -q^{-1} \int\limits_{v(a)=-1} (\pi(s_2) R W)(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a$$

$$+ (1 - q^{-1}) \int\limits_{v(a)\geq 0} (\pi(s_2) R W)(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix})|a|^{s-3/2}\, d^\times a.$$

By Corollary 7.3.3 and Proposition 7.3.2 of [RS] the first term is zero and the second term is $(1-q^{-1})Z_N(s,\pi(s_2)RW) = (1-q^{-1})Z(s,W)$. Thus,

$$B = (1-q^{-1})Z(s,W).$$

Hence,

$$Z(s,RW) = A + B = q^{-1}Z_N(s,RW) + (1-q^{-1})Z(s,W).$$

This completes the proof. □

Lemma 2.6. *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let n be a non-nonegative integer with $n \geq 2$, and let $W \in V(n)$. Then*

$$u_n\big[Z_N(s,RW)\big] = Z_N(s,R\pi(u_n)W) = Z_N(s,\pi(u_n)RW).$$

Proof. We have by Lemma 2.5 and the basic properties of $u_n\big[\,\cdot\,\big]$ from above,

$$\begin{aligned}
u_n\big[Z_N(s,RW)\big] &= u_n\big[qZ(s,RW) - (1-q^{-1})qZ(s,W)\big] \\
&= qu_n\big[Z(s,RW)\big] - (1-q^{-1})qu_n\big[Z(s,W)\big] \\
&= qZ(s,\pi(u_n)RW) - (1-q^{-1})qZ(s,\pi(u_n)W) \\
&= qZ(s,R\pi(u_n)W) - (1-q^{-1})qZ(s,\pi(u_n)W) \\
&= Z_N(s,R\pi(u_n)W) \\
&= Z_N(s,\pi(u_n)RW).
\end{aligned}$$

This completes the proof. □

Lemma 2.7. *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let n be a non-nonegative integer with $n \geq 2$, and let $W \in V(n)$. Then*

$$\begin{aligned}
u_n\big[Z_N(s,\pi(\eta^{-1})\pi(u_n)W)\big] &= Z_N(s,\pi(\eta^{-1})W) \\
&\quad + q^{-3}\big(u_n\big[Z(s,\delta_1\pi(u_n)W)\big] - Z(s,\delta_1 W)\big).
\end{aligned}$$

Proof. By Lemma 2.4,

$$Z(s,\delta_1 W) = q^3 Z_N(s,\pi(\eta^{-1})W) + Z_N(s,RW)$$

for $W \in V(n)$. Replacing W with $\pi(u_n)W$, we obtain

$$Z_N(s,\pi(\eta^{-1})\pi(u_n)W) = q^{-3}Z(s,\delta_1\pi(u_n)W) - q^{-3}Z_N(s,R\pi(u_n)W).$$

Applying $u_n\big[\,\cdot\,\big]$ to both sides and using Lemmas 2.6 and 2.4, we get

$$\begin{aligned}
&u_n[Z_N(s,\pi(\eta^{-1})\pi(u_n)W)] \\
&\quad = q^{-3}u_n\big[Z(s,\delta_1\pi(u_n)W)\big] - q^{-3}u_n\big[Z_N(s,R\pi(u_n)W)\big] \\
&\quad = q^{-3}u_n\big[Z(s,\delta_1\pi(u_n)W)\big] - q^{-3}Z_N(s,RW) \\
&\quad = q^{-3}u_n\big[Z(s,\delta_1\pi(u_n)W)\big] - q^{-3}(Z(s,\delta_1 W) - q^3 Z_N(s,\pi(\eta^{-1})W)) \\
&\quad = q^{-3}u_n\big[Z(s,\delta_1\pi(u_n)W)\big] - q^{-3}Z(s,\delta_1 W) + Z_N(s,\pi(\eta^{-1})W).
\end{aligned}$$

This completes the proof. □

To end this section we finally deduce the formula relating the second row of $m(W)$ to the first row under the assumption that $\delta_1 W = 0$ and $\delta_1 \pi(u_n)W = 0$.

Proposition 2.8. *Let (π, V) be a generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let n be a non-nonegative integer with $n \geq 2$, and let $W \in V(n)$. Assume $\delta_1 W = 0$ and $\delta_1 \pi(u_n)W = 0$. Then*

$$u_n[Z_N(s, \pi(\eta^{-1})\pi(u_n)W)] = Z_N(s, \pi(\eta^{-1})W),$$

and consequently,

$$Z_N(s, \pi(\eta^{-1})W) = (q-1)q^{-2}Z(s, W) - (q-1)q^{-2}\frac{u_n\big[(\pi(u_n)W)(1)\big](q^{-s})^2 - W(1)q^{-1}}{(q^{-s})^2 - q^{-1}}.$$

Proof. This is immediate from Lemma 2.7 and Proposition 2.3. □

3 The Alternative Proof

In this final section we will give the alternative proof of the theorem stated in the introduction. In fact, we will prove more: besides proving the claims of the theorem we will also determine $m(W_\pi)$ completely. In the preceding two sections supercuspidality was only assumed in Proposition 1.2, which asserted that $m(W)$ has only finitely many non-zero entries if W is paramodular and π is supercuspidal. We will use this below. We will also use three other properties of supercuspidal representations. Let (π, V) be a supercuspidal, generic, irreducible, admissible representation of $\mathrm{GSp}(4, F)$ with trivial central character. First, we will often use, without comment, that $Z(s, W)$ is a polynomial in q^{-s} for a paramodular vector W in V. This follows from Proposition 4.1.4 of [RS] since $L(s, \pi) = 1$. Second, we will use that the γ-factor and the ε-factor of π are the same: $\gamma(s, \pi) = \varepsilon(s, \pi)$. This follows because $L(s, \pi) = 1$. We can and will write

$$\gamma(s, \pi) = \varepsilon(s, \pi) = cq^{-Ks} \tag{2}$$

for some integer K and complex number c by Proposition 2.6.6 of [RS]. Note that, as explained in the introduction to [RS], if one has the appropriate main results of [RS], then there is a formula for $\varepsilon(s, \pi)$ in terms of the invariants of a newform, but since we are giving an alternative proof we can not use this. Third, we will use that $N_\pi \geq 2$. This is true because if $N_\pi \leq 1$, then π admits a non-zero vector fixed by the Iwahori subgroup, and is thus contained in a representation induced from the Borel subgroup. We begin with a lemma that will be applied at the minimal paramodular level N_π and at level $N_\pi + 1$.

Lemma 3.1. *Let (π, V) be a supercupsidal, generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Let n be a non-negative integer with $n \geq 2$. Assume that $W \in V(n)$ satisfies the following conditions:*

$$\delta_1 W = 0, \qquad \delta_1 \pi(u_n)W = 0, \qquad W(1) = 0.$$

Then $(\pi(u_n)W)(1) = 0$. If $V(n)$ contains no non-zero degenerate vectors, then $W = 0$.

Proof. By Proposition 2.8 and $W(1) = 0$, we have

$$Z_N(s, \pi(\eta^{-1})W) = (q-1)q^{-2}Z(s,W) - (q-1)q^{-2}\frac{u_n\big[(\pi(u_n)W)(1)\big](q^{-s})^2}{(q^{-s})^2 - q^{-1}}.$$

Therefore, by the definition of $u_n\big[\,\cdot\,\big]$,

$$\begin{aligned} &Z_N(s, \pi(\eta^{-1})W) - (q-1)q^{-2}Z(s,W) \\ &\quad = -(q-1)q^{-2} \cdot \frac{q^{n/2}(q^{-s})^{n+2}\gamma(s,\pi)^{-1}(\pi(u_n)W)(1)}{(q^{-s})^2 - q^{-1}} \\ &\quad = -(q-1)q^{-2} \cdot \frac{c^{-1}q^{n/2}(\pi(u_n)W)(1)(q^{-s})^{n-K+2}}{(q^{-s})^2 - q^{-1}}, \end{aligned}$$

Since the left hand side of this equation is a polynomial in q^{-s} by Proposition 2.1 and Proposition 1.2, so is the right hand side. Therefore, as the denominator on the right hand side has roots $\pm q^{-1/2}$, we must have $(\pi(u_n)W)(1) = 0$. Hence,

$$Z_N(s, \pi(\eta^{-1})W) = (q-1)q^{-2}Z(s,W).$$

This implies that for $k \geq 0$,

$$W(\begin{bmatrix} \varpi^{k+1} & & & \\ & \varpi^k & & \\ & & 1 & \\ & & & \varpi^{-1} \end{bmatrix}) = (q-1)q^{-2}W(\begin{bmatrix} \varpi^k & & & \\ & \varpi^k & & \\ & & 1 & \\ & & & 1 \end{bmatrix}),$$

or

$$W(\begin{bmatrix} \varpi^{2\cdot 1+k} & & & \\ & \varpi^{1+k} & & \\ & & \varpi^1 & \\ & & & 1 \end{bmatrix}) = (q-1)q^{-2}W(\begin{bmatrix} \varpi^k & & & \\ & \varpi^k & & \\ & & 1 & \\ & & & 1 \end{bmatrix}).$$

In terms of the matrix

$$m(W) = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix},$$

this means $r_1 = (q-1)q^{-2}r_0$, or equivalently,

$$r_0 + q^2r_1 - qr_0 = 0.$$

Since $\delta_1 W = 0$, we have by Corollary 1.5

$$J(m(W)) = \begin{bmatrix} r_0 + q^2r_1 \\ q^2r_2 \\ q^2r_3 \\ \vdots \end{bmatrix} \in M(n).$$

Therefore,

$$\begin{bmatrix} r_0 + q^2r_1 \\ q^2r_2 \\ q^2r_3 \\ \vdots \end{bmatrix} - q\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ q^2r_2 - qr_1 \\ q^2r_3 - qr_2 \\ \vdots \end{bmatrix}$$

is also contained in $M(n)$. Hence we produced a degenerate vector at level n. Since, by assumption, $V(n)$ has no non-zero degenerate vectors, it follows that

$$\begin{aligned} qr_1 &= q^2 r_2, \\ qr_2 &= q^2 r_3, \\ qr_3 &= q^2 r_4, \\ &\vdots \end{aligned}$$

Since π is supercuspidal we have $r_k = 0$ for sufficiently large k. This implies $0 = r_1 = r_2 = r_3 = \dots$. As $r_1 = (q-1)q^{-2} r_0$, we get $r_0 = 0$. Since $W \mapsto m(W)$ is an isomorphism, we conclude $W = 0$. □

The next theorem proves that there is uniqueness at the minimal paramodular level; this proves part of the theorem from the introduction. The remaining assertion of the theorem from the introduction will be proven in the final theorem below.

Theorem 3.2. *Let (π, V) be a supercupsidal, generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. We have:*

1. $\dim V(N_\pi) = 1$.
2. *Write $\varepsilon(s, \pi) = cq^{-Ks}$ as in (2). Then $N_\pi \geq K$ and $N_\pi \equiv K$ (2).*
3. *$V(N_\pi)$ is spanned by an element W with matrix $m(W)$ equal to*

$$\begin{bmatrix} 1 & 0 & q^{-2} & 0 & \cdots & 0 & q^{-(N_\pi - K)} & 0 & 0 & 0 & \cdots \\ -q^{-2} & 0 & -q^{-6} & 0 & \cdots & 0 & -q^{-(N_\pi - K + 2)} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

4. *Let $\pi(u_{N_\pi})W = \varepsilon_\pi W$. Then $\varepsilon_\pi = cq^{-K/2}$.*

Proof. We shall write n for N_π. As we mentioned above, since π is supercuspidal we have $n \geq 2$. Suppose that $\dim V(n) > 1$. Let $W_1, W_2 \in V(n)$ be linearly independent. There exist $a, b \in \mathbb{C}$ such that $W = aW_1 + bW_2$ is not zero and $W(1) = 0$. Since we are at the minimal level, $\delta_1 W = \delta_1 \pi(u_n) W = 0$. By the η Principle, Theorem 4.3.7 of [RS], the space $V(n)$ contains no non-zero degenerate vectors. From Lemma 3.1 we conclude $W = 0$, a contradiction. This proves $\dim V(n) = 1$.

Next, let $W \in V(n)$ be non-zero. Write

$$m(W) = \begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}$$

with

$$r_0 = (a_0, a_1, a_2, \dots),$$

$$r_1 \quad = \quad (b_0, b_1, b_2, \dots).$$

By definition,

$$\begin{aligned} Z(s,W) &= \sum_{k=0}^{\infty}(1-q^{-1})a_k q^{3k/2}(q^{-s})^k, \\ Z_N(s,\pi(\eta^{-1})W) &= \sum_{k=0}^{\infty}(1-q^{-1})b_k q^{3k/2}(q^{-s})^k. \end{aligned}$$

Also, let $\pi(u_n)W = \varepsilon_\pi W$. Similarly as in the proof of Lemma 3.1 we conclude from Proposition 2.8 that

$$\begin{aligned} &Z_N(s,\pi(\eta^{-1})W) - (q-1)q^{-2}Z(s,W) \\ &\quad = -(q-1)q^{-2}W(1)\cdot\frac{c^{-1}q^{n/2}\varepsilon_\pi(q^{-s})^{n-K+2}-q^{-1}}{(q^{-s})^2-q^{-1}}. \end{aligned} \tag{3}$$

As in the proof of Lemma 3.1, this is a polynomial in q^{-s}. It follows that $n \geq K$. Since $\pm q^{-1/2}$ are the roots of the denominator, $\pm q^{-1/2}$ are roots of the numerator. A computation shows that this implies that

$$\varepsilon = cq^{-K/2}, \quad n \equiv K\ (2).$$

Hence (3) translates into the equality

$$\begin{aligned} &\sum_{k=0}^{\infty}(1-q^{-1})(b_k-(q-1)q^{-2}a_k)q^{3k/2}(q^{-s})^k \\ &\qquad = -(q-1)q^{-2}a_0\sum_{k=0}^{(n-K)/2}q^k(q^{-s})^{2k}. \end{aligned} \tag{4}$$

Now since $\dim V(n) = 1$, there exists $a \in \mathbb{C}$ such that $J(m(W)) = am(W)$. That is,

$$J(m(W)) = \begin{bmatrix} r_0+q^2r_1 \\ q^2r_2 \\ q^2r_3 \\ \vdots \end{bmatrix} = a\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \end{bmatrix}.$$

Solving, we find that

$$r_k = q^{-2k}a^{k-1}(a-1)r_0, \quad k \geq 1.$$

Again, $r_k = 0$ for sufficiently large k. Also, $r_0 \neq 0$ since W must be nondegenerate by the η Principle. Therefore, $a = 0$ or $a = 1$. Assume $a = 1$; we will obtain a contraction. Since $a = 1$,

$$m(W) = \begin{bmatrix} r_0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

In particular, $r_1 = 0$. Therefore, from (4) we get

$$\begin{aligned} -(q-1)q^{-2}Z(s,W) &= \sum_{k=0}^{\infty} -(1-q^{-1})(q-1)q^{-2}a_k q^{3k/2}(q^{-s})^k \\ &= -(q-1)q^{-2}a_0 \sum_{k=0}^{(n-K)/2} q^k (q^{-s})^{2k}. \end{aligned}$$

Since $Z(s,W) \neq 0$, we have $a_0 \neq 0$. Comparing constant terms, we get

$$\begin{aligned} -(1-q^{-1})(q-1)q^{-2}a_0 &= -(q-1)q^{-2}a_0, \\ 1-q^{-1} &= 1, \end{aligned}$$

a contradiction. Therefore, $a = 0$. Since $a = 0$, we have

$$m(W) = \begin{bmatrix} r_0 \\ -q^{-2}r_0 \\ 0 \\ \vdots \end{bmatrix},$$

i.e.,

$$b_k = -q^{-2}a_k, \quad k \geq 0.$$

Therefore, we get from (4) that

$$\sum_{k=0}^{\infty} q^{3k/2} a_k (q^{-s})^k = a_0 \sum_{k=0}^{(n-K)/2} q^k (q^{-s})^{2k}.$$

We obtain $a_0 \neq 0$. Dividing if necessary, we may assume that $a_0 = 1$. Therefore,

$$a_i = \begin{cases} 0 & \text{if } i \text{ is odd or } i > n-K, \\ q^{-i} & \text{if } i \text{ is even and } 0 \leq i \leq n-K. \end{cases}$$

The remaining claims of the theorem follow. □

Lemma 3.3. *Let (π, V) be a supercupsidal, generic, irreducible, admissible representation of* $\mathrm{GSp}(4,F)$ *with trivial central character, and let $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. Then* $\dim V(N_\pi + 1) \leq 3$.

Proof. For convenience, write $n = N_\pi$. By Theorem 3.2 we have $\dim V(n) = 1$. Choosing any isomorphism $V(n) \cong \mathbb{C}$, we can consider $\delta_1 : V(n+1) \to V(n)$ as a linear form on $V(n+1)$. We consider further the linear forms $\delta_1 \circ \pi(u_{n+1})$ and $\varphi : W \mapsto W(1)$ on $V(n+1)$. Let $W \in V(n+1)$ and assume that

$$W \in \ker(\delta_1) \cap \ker(\delta_1 \circ \pi(u_{n+1})) \cap \ker(\varphi).$$

In other words, W is an element such that $\delta_1 W = 0$ and $\delta_1 \pi(u_{n+1})W = 0$ and $W(1) = 0$. Lemma 3.1 implies that $W = 0$; note that $V(n+1)$ contains no degenerate vectors by the η Principle from [RS]. This shows that $\ker(\delta_1) \cap \ker(\delta_1 \circ \pi(u_{n+1})) \cap \ker(\varphi) = 0$. On the other hand,

$$\dim(\ker(\delta_1) \cap \ker(\delta_1 \circ \pi(u_{n+1})) \cap \ker(\varphi)) \geq \dim(V(n+1)) - 3,$$

since with every linear form the dimension can go down by at most one. The assertion follows. □

Theorem 3.4. *Let (π, V) be a supercupsidal, generic, irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *with trivial central character, and let* $V = \mathcal{W}(\pi, \psi_{c_1,c_2})$. *The newform in Theorem 3.2 iii) is given by*

$$m(W) = \begin{bmatrix} 1 & 0 & \cdots \\ -q^{-2} & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \end{bmatrix}.$$

Proof. Again for convenience we let $n = N_\pi$. Let W_0 be the vector in Theorem 3.2. By this theorem, we have

$$m(W_0) = \begin{bmatrix} s_0 \\ -q^{-2}s_0 \\ 0 \end{bmatrix}, \quad s_0 = (1,\, 0,\, q^{-2},\, 0,\, q^{-4},\, \ldots,\, q^{-(n-K)},\, 0,\, 0,\, \ldots)$$

(all the matrices in this proof will have zeros in the fourth row and beyond, hence we shall only write the first three rows). By Proposition 1.3 we have

$$m(\theta' W_0) = \begin{bmatrix} qs_0 \\ (1-q^{-1})s_0 \\ -q^{-2}s_0 \end{bmatrix} \tag{5}$$

and

$$m(\theta W_0) = q \begin{bmatrix} 0 \\ \mathrm{Left}(s_0) \\ \mathrm{Left}(-q^{-2}s_0) \end{bmatrix} + \begin{bmatrix} \mathrm{Right}(s_0) \\ \mathrm{Right}(-q^{-2}s_0) \\ 0 \end{bmatrix}. \tag{6}$$

Define $W_1 := q^{-2}\delta_1\theta'\theta W_0 - q\theta W_0 \in V(n+1)$. By Lemma 1.5 we have

$$q^{-2}m(\delta_1\theta' W) - qm(W) = J(m(W)) + q^{-2}m(\eta\delta_1 W)$$

for any paramodular vector W. Applying this with $W = \theta W_0$ we get

$$m(W_1) = J(m(\theta W_0)) + q^{-2}m(\eta\delta_1\theta W_0).$$

Since $\dim(V(n)) = 1$, we have $\delta_1\theta W_0 = \alpha W_0$ for some $\alpha \in \mathbb{C}$ (which might be zero). Hence

$$\begin{aligned} m(W_1) &= J(m(\theta W_0)) + \alpha q^{-2}m(\eta W_0) \\ &= J(q \begin{bmatrix} 0 \\ \mathrm{Left}(s_0) \\ \mathrm{Left}(-q^{-2}s_0) \end{bmatrix}) + \alpha q^{-2}m(\eta W_0) \\ &= \begin{bmatrix} q^3\mathrm{Left}(s_0) \\ -q\mathrm{Left}(s_0) \\ 0 \end{bmatrix} + \alpha q^{-2} \begin{bmatrix} 0 \\ s_0 \\ -q^{-2}s_0 \end{bmatrix}. \end{aligned} \tag{7}$$

Let us now assume that W_0 does *not* have the asserted form; we shall derive a contradiction. Thus we assume that $n > K$, or equivalently, that $\mathrm{Left}(s_0) \neq 0$. Under this assumption we have $W_1 \neq 0$. In fact, it is easy to see that the matrices given in (5), (6) and (7) are linearly independent. By Lemma 3.3 we get $\dim(V(n+1)) = 3$ and

$$V(n+1) = \langle \theta' W_0,\, \theta W_0,\, W_1 \rangle. \tag{8}$$

Now consider the vector $W_2 := q\theta W_0 - W_1$. The first row of $m(W_2)$ is given by

$$q\mathrm{Right}(s_0) - q^3\mathrm{Left}(s_0) = (0, \ldots, 0, q^{-(n-K)+1}, 0, \ldots),$$

where the non-zero entry is at position $n - K + 1$ (the first entry is at position 0). Therefore

$$Z(s, W_2) = \text{const.} \cdot (q^{-s})^{n-K+1}.$$

By the functional equation we have

$$Z(s, \pi(u_{n+1})W) = q^{-(n+1)s} q^{(n+1)/2} \gamma(1 - s, \pi) Z(1 - s, W)$$

for any $W \in V(n+1)$. Applied to $W = W_2$ we get

$$\begin{aligned} Z(s, \pi(u_{n+1})W_2) &= q^{-(n+1)s} q^{(n+1)/2} \gamma(1 - s, \pi) Z(1 - s, W_2) \\ &= \text{const.} \cdot q^{-(n+1)s} \gamma(1 - s, \pi)(q^{-(1-s)})^{n-K+1} \\ &= \text{const.} \cdot q^{-(n+1)s} \gamma(1 - s, \pi)(q^{s})^{n-K+1} \\ &= \text{const.} \cdot q^{-(n+1)s} \varepsilon_\pi q^{-K/2} (q^{-s})^{-K} q^{s(n-K+1)} \\ &= \text{const.} \end{aligned}$$

For the fourth equality we used the fourth assertion of Theorem 3.2. Therefore, $\pi(u_{n+1})W_2 \in V(n+1)$ is a vector with constant zeta polynomial. On the other hand, by (8), there exist $x, y, z \in \mathbb{C}$ such that $\pi(u_{n+1})W_2 = x\theta' W_0 + y\theta W_0 + zW_1$. Then

$$Z(s, \pi(u_{n+1})W_2) = x \underbrace{Z(s, \theta' W_0)}_{\text{even}} + y \underbrace{Z(s, \theta W_0)}_{\text{odd}} + z \underbrace{Z(s, W_1)}_{\text{odd}}.$$

The "even" and "odd" refer to the powers of q^{-s} occuring in these zeta polynomials. Since, by (5), (6) and (7), the function $Z(s, \theta W_0)$ has higher degree in q^{-s} than the other two zeta functions, it follows that $y = 0$. Then it follows that $z = 0$ since the result must be constant and $Z(s, W_1)$ has only odd degrees. It follows that $W_2 = x\theta' W_0$. But this is impossible since the first row of $m(\theta' W_0)$ has more than one non-zero entry by our assumption. $\square$

It is evident that the claims of the theorem from the introduction follow from Theorem 3.2 and Theorem 3.4.

References

[RS] Roberts, B., Schmidt, R.: Local Newforms for GSp(4). Preprint, 305 pp. (2006)