EHRHART POLYNOMIAL REVISITED

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Let M be a lattice of rank n. Let P be a convex lattice polytope in the vector space $M_{\mathbb{R}} = M \otimes \mathbb{R}$. The number of lattice points contained in P is denoted by $\#(P)$. The Ehrhart polynomial $Ehr_P(\nu)$ of P is a polynomial in ν of degree n such that, for any positive integer ν, it coincides with the number $\#(\nu P)$, where $\nu P = \{\nu u \mid u \in P\}$. We put

$$Ehr_P(\nu) = \sum_{k=0}^{n} a_k(P)\nu^{n-k}.$$

According to the theory of toric variety a convex lattice polytope P determines a complete toric variety X and an invariant Cartier divisor D on X. If Δ is the fan in $N_{\mathbb{R}}$ (dual of $m_{\mathbb{R}}$) ssociated to X, then each k-dimensional cone σ in Δ together with D determines an $(n-k)$-dimensional face P_σ of P.

It is known that $a_k(P)$ can be written in the form

$$a_k(P) = \sum_{\dim \sigma = k} \mu_k(\sigma) \operatorname{vol} P_\sigma.$$

There is a question posed by Danilov. Can $\mu_k(\sigma)$ be chosen so that it depends only on the position of σ in $N_{\mathbb{R}}$?

There is an answer given by Morelli. Let $\mathrm{Rat}(G_{n-k+1})_0$ denote the rational functions of degree 0 of the Grassmann manifold of $(n-k+1)$-planes in $N_{\mathbb{R}}$.

Theorem 0.1 (Morelli). *There is an element $\mu_k(\sigma) \in \mathrm{Rat}(G_{n-k+1})_0$ which depends only on the position of σ in $N_{\mathbb{R}}$, and such that*

$$\sum_{\dim \sigma = k} \mu_k(\sigma)(E) \operatorname{vol} P_\sigma = a_k(P)$$

for any generic $(n-k+1)$-plane in $N_{\mathbb{R}}$.

This theorem implies that

$$\sum_{\dim \sigma = k} \mu_k(\sigma) \operatorname{vol} P_\sigma = a_k(P)$$

is a constant funciotn in $\mathrm{Rat}(G_{n-k+1})_0$.

Morelli applies Baum-Bott's localization theorem for holomorphic foliations to the foliation determined by a generic plane E and constructs $\mu_k(\sigma)$ explicitly.

I will explain Morelli's formula from a broader point of view, and give a proof without the use of the Baum-Bott theorem.