Constructions of three-dimensional small covers

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Small covers were introduced by Davis and Januszkiewicz [1] as n-dimensional closed manifolds M^n with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is a simple convex polytope P where $\mathbb{Z}_2$ is the quotient additive group $\mathbb{Z}/2\mathbb{Z}$. They showed that there exists a one-to-one correspondence between small covers and $(\mathbb{Z}_2)^n$-colored polytopes (cf. [1, Proposition 1.8]). In this talk we are interested in constructions of 3-dimensional small covers M^3 by using operations called a *connected sum* $\sharp$ and a *surgery* $\natural$. In [2] Izmestiev studied a class of 3-dimensional small covers which are called *linear models* and are correspondent to 3-colored polytopes, and proved the following theorem (cf. [2, Theorem 3]).

Theorem 1.1 (Izmestiev) *Each linear model M^3 can be constructed from the 3-dimensinal torus T^3 by using three operations $\sharp$, $\natural$ and $\natural^{-1}$.*

In [5] we generalized the above theorem to orientable small covers. Later Lü and Yu [4] considered a construction of general 3-dimensional small covers. They introduced new operations $\sharp^e$, $\sharp^{eve}$, $\sharp^\Delta$ and $\sharp_i^C$ and showed the following theorem (cf. [4, Theorem 1.2]).

Theorem 1.2 (Lü and Yu) *Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ by using seven operations $\sharp$, $\natural^{-1}$, $\sharp^e$, $\sharp^{eve}$, $\sharp^\Delta$, $\sharp_4^C$ and $\sharp_5^C$.*

Operations appeared in Theorem 1.2 are all "non-decreasing" unlike Theorem 1.1, and therefore the use of the surgery $\natural$ is prohibited. In [3, Theorem 4.1] Kuroki pointed out that $\sharp^e$ and $\sharp^{eve}$ can be obtained as compositions of $\natural$ and $\sharp$, and so Theorem 1.2 can be rewritten as follows: Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ by using six operations $\sharp$, $\natural$, $\natural^{-1}$, $\sharp^\Delta$, $\sharp_4^C$ and $\sharp_5^C$ (cf. [3, Corollary 4.8]). Then the following is our main result in this talk.

Theorem 1.3 *(1) Each small cover M^3 can be constructed from T^3, $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using two operations $\sharp$ and $\natural$.*
(2) Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using four operations $\sharp$, $\sharp^e$, $\natural^{-1}$ and $\sharp_4^C$.

References

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