On projective bundles over small covers

Shintarô Kuroki (KAIST)
with Zhi Lü (Fudan University)

The 36th Symposium on Transformation Groups (2009)
Osaka City University

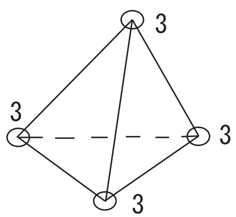
Contents

1. Basic facts of small covers

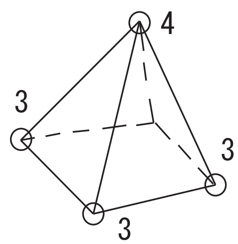
Small cover (Davis-Januszkiewicz, 1990)
$\overset{def}{\Longleftrightarrow}$ a compact n-dimensional manifold M^n with the following two conditions:

1. M^n has an effective, locally standard $(\mathbf{Z}_2)^n$-action, i.e., locally looks like the standard $(\mathbf{Z}_2)^n \curvearrowright \mathbf{R}^n$;

2. the orbit space is an n-dimensional simple polytope $M^n/(\mathbf{Z}_2)^n = P^n$, i.e., each vertex is constructed by the intersection of just <u>n facets</u>.



Simple polytope



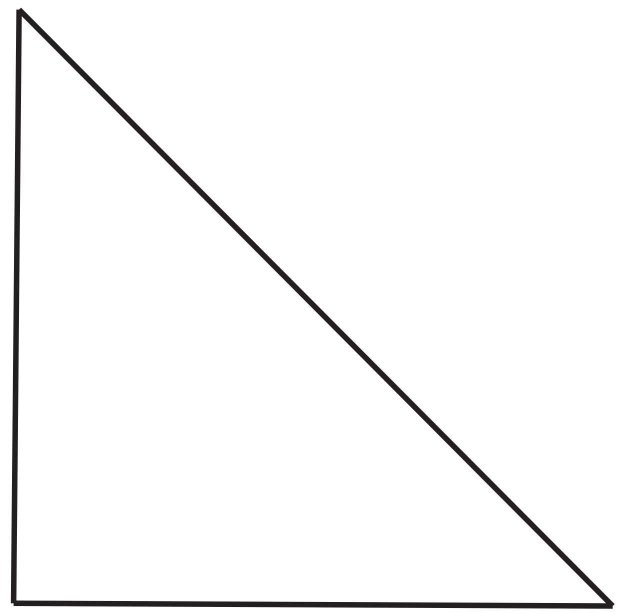
Non-simple polytope

Example

Example 1. *Let* $\mathbf{R}P(n)$ *be the* n*-dimensional real projective space with the following* $\mathbf{Z}_2^n$*-action:*

$$\begin{aligned} & (t_1, \ldots, t_n) \cdot [r_0 : r_1 : \cdots : r_n] \\ = \quad & [r_0 : t_1 r_1 : \cdots : t_n r_n]. \end{aligned}$$

Then this action is locally standard and $\mathbf{R}P(n)/\mathbf{Z}_2^n = \Delta^n$.



$\mathbf{R}P(2)/\mathbf{Z}_2^2 = \Delta^2$.

Two datas of small covers

From the small cover, we have the following two datas.

1. P^n: an n-dimensional simple polytope,

2. $\lambda : \mathcal{F} \to \{0, 1\}^n$: a characteristic function, that is,

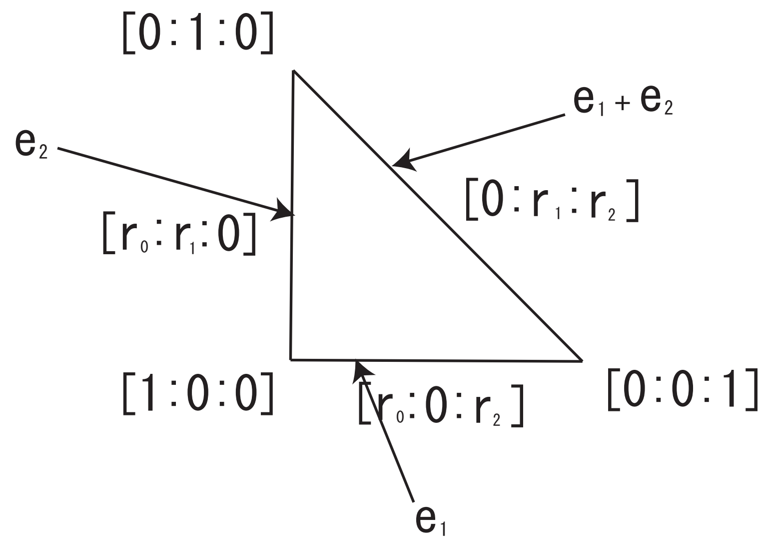
$$\det(\lambda(F_1), \ldots, \lambda(F_n)) = 1 \pmod 2 \text{ for } \cap_{i=1}^n F_i = \{v\},$$

where $\mathcal{F} = \{F_1, \ldots, F_m\}$ denotes the set of all facets (codimension-one faces) of P.

Characteristic function of $\mathbf{R}P(2)$

Let $\mathbf{R}P(n)$ be the n-dimensional real projective space with the following $\mathbf{Z}_2^n$-action:

$$(t_1, t_2) \cdot [r_0 : r_1 : r_2] = [r_0 : t_1 r_1 : t_2 r_2].$$



The isotropy group of $[r_0 : 0 : r_2]$ is $\mathbf{Z}_2 \times \{1\} \Longrightarrow e_1 \in \{0,1\}^2$.

The isotropy group of $[r_0 : r_1 : 0]$ is $\{1\} \times \mathbf{Z}_2 \Longrightarrow e_2 \in \{0,1\}^2$.

The isotropy group of $[0 : r_1 : r_2]$ is $\Delta \Longrightarrow e_1 + e_2 \in \{0,1\}^2$.

Characterization of small covers

Small covers can be reconstructed from the two datas (P^n, λ).

1. P^n: an n-dimensional simple polytope,

2. $\lambda : \mathcal{F} \to \{0,\ 1\}^n$: a characteristic function.

Then

$$M(P, \lambda) = (\mathbf{Z}_2)^n \times P^n / \sim_\lambda$$

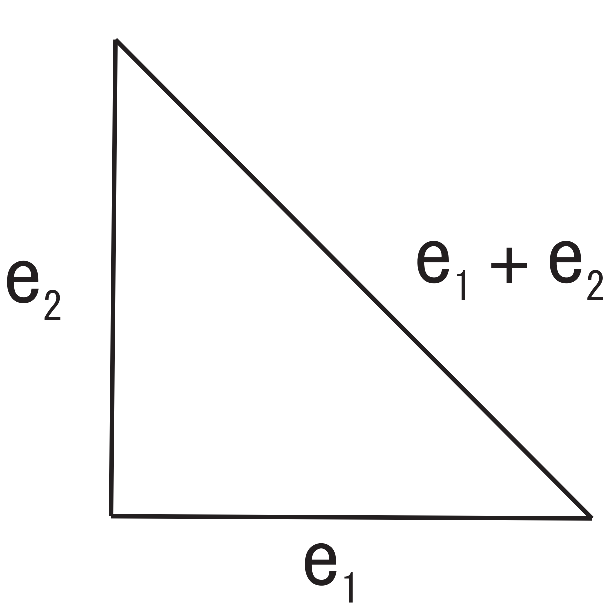
is small cover, where

$(t, p) \sim_\lambda (t', q) \quad \Leftrightarrow \quad p = q,$ and $t't^{-1} \in T(p) = \langle (\text{-}\mathbf{1})^{\lambda(F)} \mid p \in F \rangle \subset (\mathbf{Z}_2)^n$.

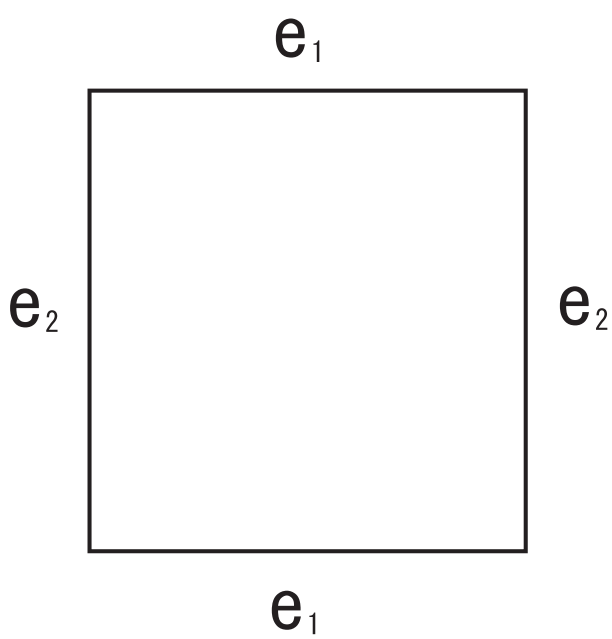
Here, $\text{-}\mathbf{1} = (-1, \ldots, -1) \in \mathbf{Z}_2^n$.

Examples

Example 2. *In the following figures, the left and right pair are called by* (Δ^2, λ_2) *and* (I^2, λ_1^2) *respectively (where* e_1 *and* e_2 *are standard basis in* $\{0,1\}^2$*).*



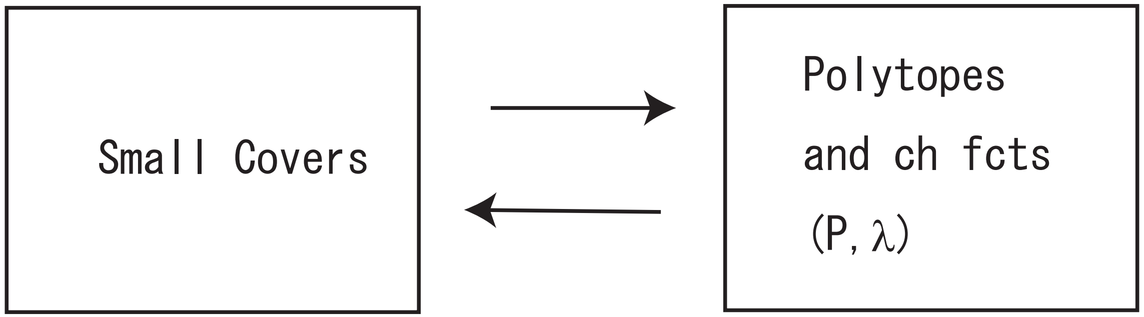
$$M(\Delta^2, \lambda_2) = \mathbf{R}P(2)$$



$$M(I^2, \lambda_1^2) = T^2$$

Summary

In summary we have the following correspondence.



The function λ is also denoted by the following matrix

$$(\lambda(F_1), \dots, \lambda(F_m)) = \begin{pmatrix} I_n & \Lambda \end{pmatrix} \in M(n, m; \mathbf{Z}_2),$$

where $\Lambda \in M(n, m-n; \mathbf{Z}_2)$. We call $(I_n\ \Lambda)$ a <u>characteristic matrix</u>.

2. Motivation

Cohomological rigidity problem for small cover

Assume $H^*(M;\mathbf{Z}_2) \simeq H^*(M';\mathbf{Z}_2)$ for two small covers M and M'.

Problem: Are M and M' homeomorphic?

$\Downarrow$

Answer: No!

There are counter examples in the above class.

Masuda's counter examples

$M(q) = P(q\gamma \oplus (b-q)\epsilon)$: the projective bundle over $\mathbf{R}P(a)$, where γ is the canonical line bundle, ϵ is the trivial bundle and $0 \leq q \leq b$.

Theorem 1 (Masuda). *The following two statements hold:*

1. $H^*(M(q); \mathbf{Z}_2) \simeq H^*(M(q'); \mathbf{Z}_2) \overset{\text{iff}}{\Leftrightarrow} q' \equiv q$ *or* $b - q \mod 2^{h(a)}$, *where* $h(a) = \min\{n \in \mathbb{N} \cup \{0\} \mid 2^n \geq a\}$;

2. $M(q) \cong M(q') \overset{\text{iff}}{\Leftrightarrow} q' \equiv q$ *or* $b - q \mod 2^{k(a)}$, *where* $k(a) = \#\{n \in \mathbb{N} \mid 0 < n < a \text{ and } n \equiv 0, 1, 2, 4 \mod 8\}$.

Put $a = 10$, then we have $h(10) = 4$, $k(10) = 5$.

Put $b = 17$ and $q = 1$ and $q' = 0$.

Then $H^*(M(1)) \simeq H^*(M(0))$ (by $q' \equiv 17 - q \mod 2^{h(10)} = 16$), but $M(1) \not\cong M(0)$ (by $q' \not\equiv 17 - q \mod 2^{k(10)} = 32$).

Problem: **Characterize (or classify)** **the topological types of projective bundles over small covers.**

3. Projective bundles over small covers

Let $\xi = (E(\xi), \pi, M, \mathbf{R}^k)$ be an equivariant k-dimensional vector bundle over a small cover M^n.

Put $\sigma_0(M)$ is the image of the zero section and

$$P(\xi) = E(\xi) - \sigma_0(M)/\mathbf{R}^*,$$

then $P(\xi)$ is the $\mathbf{R}P^{k-1}$-bundle over M.

Lemma 1. *$P(\xi)$ is a small cover $\overset{\text{iff}}{\Leftrightarrow}$ $\xi \equiv \gamma_1 \oplus \cdots \oplus \gamma_k$ where γ_i is a line bundle.*

We call such $P(\xi)$ a projective bunlde over small cover (or projective bunlde).

Lemma 2. $P(\xi)$ *has the following two properties:*

1. *the orbit space is* $P^n \times \Delta^{k-1}$ *(where* $M/\mathbf{Z}_2^n = P^n$*);*

2. *the characteristic matrix of* $P(\xi)$ *can be denoted by*

$$\begin{pmatrix} I_n & O & \Lambda & \mathbf{0} \\ O & I_{k-1} & \Lambda' & \mathbf{1} \end{pmatrix}$$

Therefore, in order to consider the projective bundle over small cover, we may only consider the following matrix:

$$\begin{pmatrix} I_n & \Lambda \\ O & \Lambda' \end{pmatrix} \in M(n+k-1, m; \mathbf{Z}_2)$$

Characterization of projective bundles

Idea: Attach this matrix to the facets of P^n directly

For example, for $\mathbf{r} = (r_1, \ldots, r_{k-1}) \in \{0,1\}^{k-1}$,

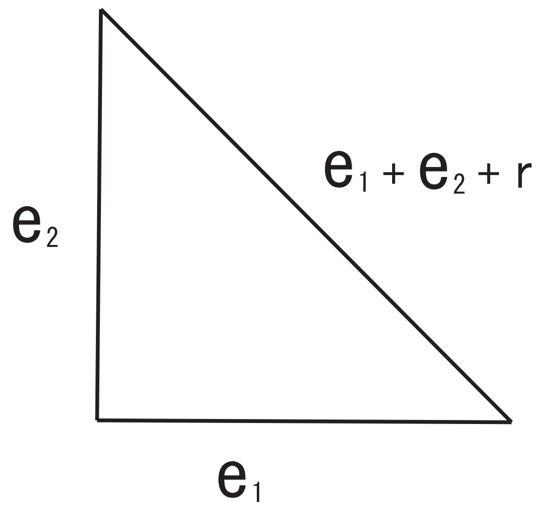
The following matrix

$$\begin{pmatrix} I_2 & \mathbf{1} \\ O & \mathbf{r} \end{pmatrix} \in M(k+1, 3; \mathbf{Z}_2),$$

corresponds with

$$P(\gamma^{r_1} \oplus \cdots \oplus \gamma^{r_{k-1}} \oplus \epsilon),$$

where $\gamma^0 = \epsilon$ and $\gamma^1 = \gamma$ over $\mathbf{R}P(2)$.



$\mathbf{r} = r_1 e_1' + \cdots + r_{k-1} e_{k-1}'$.

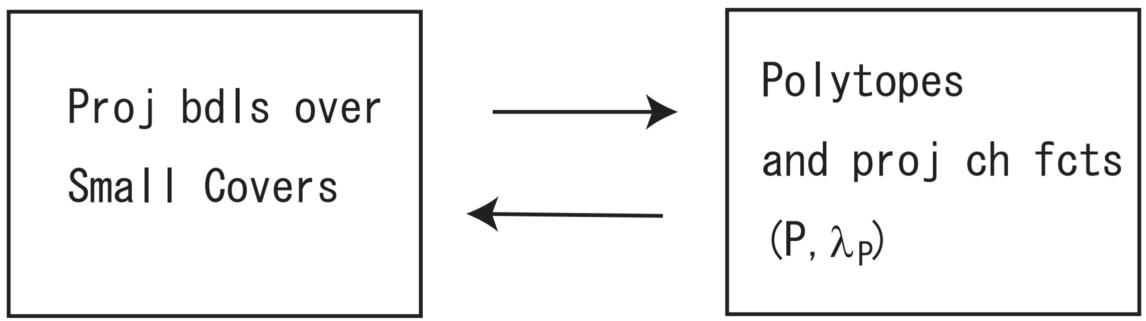
Projective characteristic functions

$\lambda_P : \mathcal{F}_P \to \{0,1\}^n \times \{0,1\}^{k-1}$: <u>**projective characteristic functions**</u> such that

$$\det(\lambda_P(F_{i_1}), \ldots, \lambda_P(F_{i_n}), X_1 \ldots, X_{k-1}) = 1$$

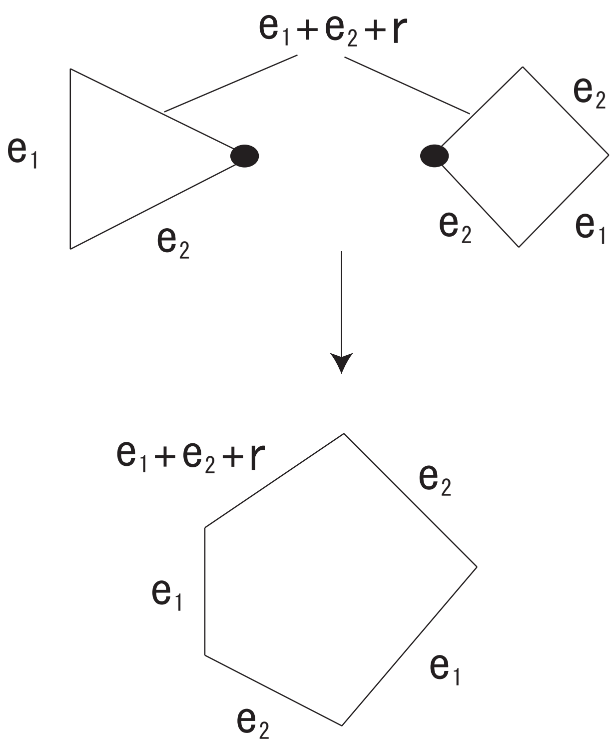
for $F_{i_1} \cap \cdots \cap F_{i_n} \neq \emptyset$ and $\{X_1, \ldots, X_{k-1}\} \subset \{e'_1, \ldots, e'_{k-1}, \mathbf{1}\}$, where e'_i is the standard basis of $\{0,1\}^{k-1}$.

Then <u>(P, λ_P) characterizes the projective bundle over small cover</u>.



New operation on projective characteristic functions

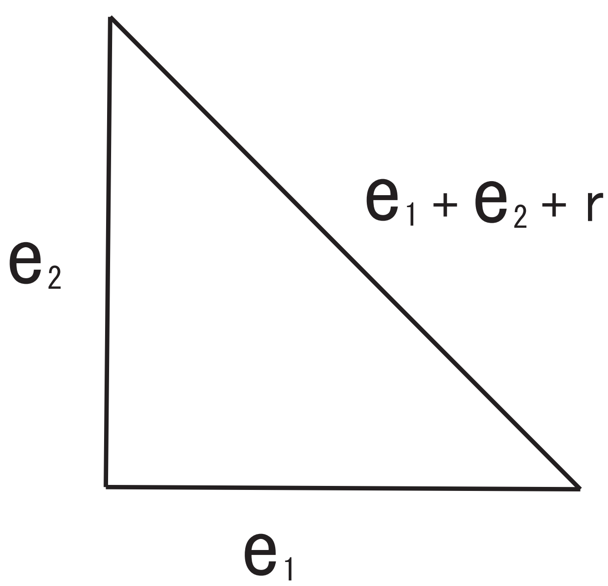
In order to prove the construction theorem of projective bundles over 2-dim small covers, we introduce a operation $\sharp^{\Delta^{k-1}}$ on the projective characteristic functions as follows.

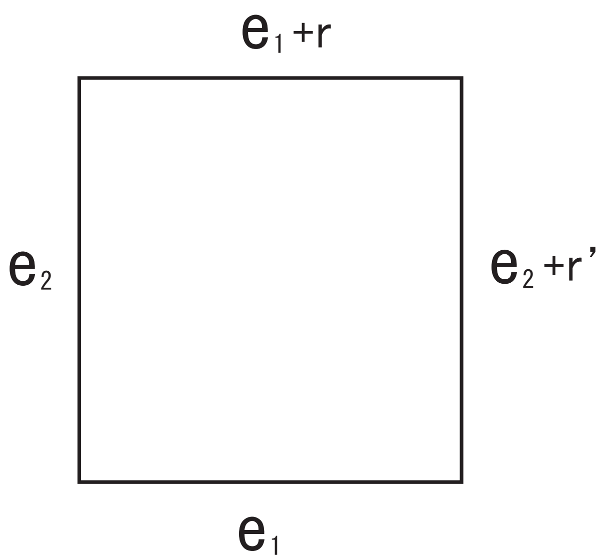


Remark: This operation corresponds with the fibre some of projective bundles (gluing along the fibres).

Main Theorem

Theorem 2. *Let $P(\xi)$ be a projective bundle over 2-dimensional small cover M^2. Then $P(\xi)$ can be constructed from projective bundles $P(\zeta)$ over the real projective space $\mathbf{R}P^2$ and $P(\kappa)$ over the torus T^2 by using $\sharp^{\Delta^{k-1}}$.*





$$P(\zeta) = P(\gamma^{r_1} \oplus \cdots \oplus \gamma^{r_{k-1}} \oplus \epsilon)$$

$$P(\kappa) = P(\gamma^{r_1} \otimes \gamma^{r'_1} \oplus \cdots \oplus \gamma^{r_{k-1}} \otimes \gamma^{r'_{k-1}} \oplus \epsilon)$$

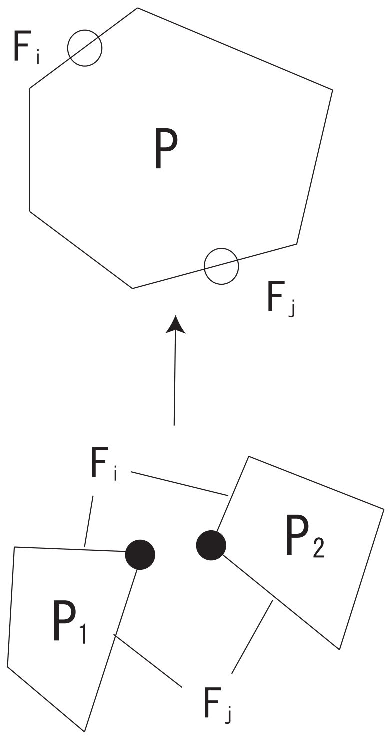
Outline of proof

Step 1: Prove there are two edges F_i, F_j such that

$$\det(\lambda_P(F_i), \lambda_P(F_j), X_1, \ldots, X_{k-1}) = 1.$$

Step 2: Then we can do the converse of the operation $\sharp^{\Delta^{k-1}}$ along F_i and F_j.

Step 3: Iterating the above argument, finally P decomposes into the sum of Δ^2's and I^2's.



Step 1 and **Step 2**

Topological classification of the basic parts

Finally we list up all topological types of projective bundles over $\mathbf{R}P(2)$ and T^2.

Proposition 1. *The topological type of $P(\zeta)$ is one of the following 4 topological types:*

$$S^2 \times_{\mathbf{Z}_2} P(q\mathbf{R} \oplus (k-q)\underline{\mathbf{R}}),$$

for $q = 0, 1, 2, 3$.

Proposition 2. *The topological type of $P(\kappa)$ is one of the following 4 topological types:*

$$T^2 \times_{\mathbf{Z}_2^2} P(\mathbf{R}_1 \oplus \mathbf{R}_2 \oplus (k-2)\underline{\mathbf{R}});$$
$$T^2 \times_{\mathbf{Z}_2^2} P(\mathbf{R}_1 \oplus (k-1)\underline{\mathbf{R}});$$
$$T^2 \times_{\mathbf{Z}_2^2} P(\mathbf{R}_2 \oplus (k-1)\underline{\mathbf{R}});$$
$$T^2 \times \mathbf{R}P(k-1),$$

where $T^2 \times_{\mathbf{Z}_2} \mathbf{R}_i$ is the canonical bundle of the i-th $S^1 \subset T^2$ ($i = 1, 2$).