Möbius transform, moment-angle complexes and Halperin-Carlsson conjecture

—A joint work with Xiangyu Cao

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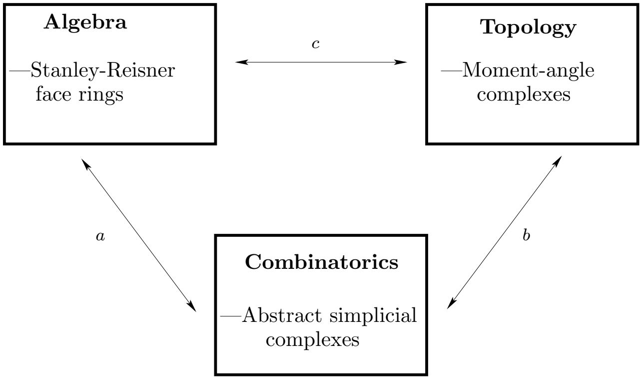
§1 Background—A triangle



References

- For the edge a, see
  [1] Stanley, Richard P, *Combinatorics and commutative algebra. Second edition*, Progress in Mathematics, **41**, Birkhäuser Boston, Inc., Boston, MA, 1996.
  [2] E. Miller and B. Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Math. **227**, Springer, 2005.
- For other two edges b, c, see
  [3] V. M. Buchstaber and T. E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, Vol. **24**, Amer. Math. Soc., Providence, RI, 2002.

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Notion—Abstract simplicial complex

Let $[m] = \{1, ..., m\}$.

–Abstract simplicial complexes on $[m]$

- **An abstract simplicial complex K on $[m]$** is a collection of some subsets in $[m]$ such that for each $a \in K$, any subset (including $\emptyset$) of a belongs to K.
- Each a in K is called a simplex of dim$= |a| - 1$, and $\dim K = \max_{a \in K}\{\dim a\}$.

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Notion—Stanley-Reisner face ring

K: an abstract simplicial complex on $[m]$

$\mathbf{k}$: a field.

Stanley-Reisner face ring

$$\mathbf{k}(K) = \mathbf{k}[v_1, ..., v_m]/I_K$$

is called the **Stanley-Reisner face ring** of K, and I_K is the ideal generated by all square-free monomials $v_{i_1} \cdots v_{i_s}$ with $\sigma = \{i_1, ..., i_s\} \notin K$.

RK: write $\mathbf{k}[\mathbf{v}] = \mathbf{k}[v_1, ..., v_m]$.

Notion—Betti numbers of Stanley-Reisner face ring $\mathbf{k}(K)$

It is well-known that $\mathbf{k}(K)$ is a finitely generated $\mathbb{N}^m$-graded $\mathbf{k}[\mathbf{v}]$-module and it has an minimal free resolution

$$0 \longleftarrow \mathbf{k}(K) \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \longleftarrow \cdots \longleftarrow F_{h-1} \xleftarrow{\phi_h} F_h \longleftarrow 0 \quad (1)$$

Write $F_i = \bigoplus_{\mathbf{a}\in\mathbb{N}^m} \big(\underbrace{\mathbf{k}[\mathbf{v}](-\mathbf{a}) \oplus \cdots \oplus \mathbf{k}[\mathbf{v}](-\mathbf{a})}_{\beta_{i,\mathbf{a}}^{\mathbf{k}(K)}}\big)$ where $\mathbf{k}[\mathbf{v}](-\mathbf{a})$ is the ideal $\langle \mathbf{v}^{\mathbf{a}} \rangle$, and $\mathbf{v}^{\mathbf{a}} = v_1^{a_1} \cdots v_m^{a_m}$ for $\mathbf{a} = (a_1, ..., a_m) \in \mathbb{N}^m$.

Betti number

$\beta_{i,\mathbf{a}}^{\mathbf{k}(K)} \in \mathbb{N}$ is called the $(i, \mathbf{a})$-th **Betti number of $\mathbf{k}(K)$**.

Notion—Tor-algebra of Stanley-Reisner face ring $\mathbf{k}(K)$

Applying the functor $\otimes_{\mathbf{k}[\mathbf{v}]}\mathbf{k}$ to the sequence (1) above, one may obtain the following chain complex of $\mathbb{N}^m$-graded $\mathbf{k}[\mathbf{v}]$-modules:

$$0 \longleftarrow F_0 \otimes_{\mathbf{k}[\mathbf{v}]} \mathbf{k} \xleftarrow{\phi_1'} F_1 \otimes_{\mathbf{k}[\mathbf{v}]} \mathbf{k} \longleftarrow \cdots \xleftarrow{\phi_h'} F_h \otimes_{\mathbf{k}[\mathbf{v}]} \mathbf{k} \longleftarrow 0.$$

Define $\mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K), \mathbf{k}) := \frac{\ker \phi_i'}{\mathrm{Im}\phi_{i+1}'} = F_i \otimes_{\mathbf{k}[\mathbf{v}]} \mathbf{k}$ so

$$\dim_{\mathbf{k}} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K), \mathbf{k}) = \mathrm{rank} F_i = \sum_{\mathbf{a}\in\mathbb{N}^m} \beta_{i,\mathbf{a}}^{\mathbf{k}(K)}.$$

Tor-algebra

$$\mathrm{Tor}^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K), \mathbf{k}) = \bigoplus_{i=0}^{h} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K), \mathbf{k}) = \bigoplus_{\substack{i\in[0,h]\cap\mathbb{N} \\ \mathbf{a}\in\mathbb{N}^m}} \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K), \mathbf{k})_{\mathbf{a}}$$

A remark

It is well-known that if $\mathbf{a} \in \mathbb{N}^m$ is not a vector in $\{0,1\}^m$, then $\mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K), \mathbf{k})_{\mathbf{a}} = 0$, so $\beta_{i,\mathbf{a}}^{\mathbf{k}(K)} = 0$.

$$\{0,1\}^m \longleftrightarrow 2^m$$

$$\Downarrow$$

write

$$\beta_{i,a}^{\mathbf{k}(K)} := \beta_{i,\mathbf{a}}^{\mathbf{k}(K)}$$

where $2^{[m]} \ni a \longleftrightarrow \mathbf{a} \in \{0,1\}^m$.

Moment-angle complex

A general construction

K: a simplicial complex on vertex set $[m] = \{1, ..., m\}$
(X, W): a pair of top. spaces with $W \subset X$.

$$K(X, W) := \bigcup_{\sigma \in K} \Big(\prod_{i \in \sigma} X \times \prod_{i \notin \sigma} W\Big) \subseteq X^m.$$

- $\mathcal{Z}_K := K(D^2, S^1) \subset (D^2)^m$ is called the *moment-angle complex* on K.
- $\mathbb{R}\mathcal{Z}_K := K(D^1, S^0) \subset (D^1)^m$ is called the *real moment-angle complex* on K.

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Actions on $\mathcal{Z}_K$ and $\mathbb{R}\mathcal{Z}_K$

A canonical action on $\mathcal{Z}_K$

$D^2 = \{z \in \mathbb{C} \big| |z| \leq 1\}$ and $S^1 = \partial D^2$.
Since $(D^2)^m \subset \mathbb{C}^m$ is invariant under the standard action of T^m on $\mathbb{C}^m$ given by

$$((g_1, ..., g_m), (z_1, ..., z_m)) \longmapsto (g_1 z_1, ..., g_m z_m),$$

$(D^2)^m$ admits a natural T^m-action whose orbit space is the unit cube $I^m \subset \mathbb{R}^m_{\geq 0}$. The action $T^m \curvearrowright (D^2)^m$ then induces a canonical T^m-action Φ on $\mathcal{Z}_K$.

Similarly

A canonical action on $\mathbb{R}\mathcal{Z}_K$

$\mathbb{R}\mathcal{Z}_K$ admits a canonical $(\mathbb{Z}_2)^m$-action $\Phi_{\mathbb{R}}$ on $\mathbb{R}\mathcal{Z}_K$

Hochster Theorem

On the edge a of the triangle, there is the following essential result:

Hochster Theorem

For each $a \in 2^{[m]}$,

$$\widetilde{H}^{|a|-i-1}(K|_a; \mathbf{k}) \cong \mathrm{Tor}_i^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K), \mathbf{k})_a$$

where $K|_a = \{\sigma \in K \big| \sigma \subseteq a\}$.

Buchstaber-Panov Theorem

On the edge c of the triangle, there is the following essential result:

Buchstaber-Panov Theorem

As $\mathbf{k}$-algebras,

$$H^*(\mathcal{Z}_K; \mathbf{k}) \cong \mathrm{Tor}^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K), \mathbf{k})$$

where $\mathbf{k}(K) = \mathbf{k}[\mathbf{v}]/I_K = \mathbf{k}[v_1, ..., v_m]/I_K$ with $\deg v_i = 2$, and $\mathbf{k}$ is a field.

Further development—A viewpoint of analysis

- Let $2^{[m]*}=\{f \big| f: 2^{[m]} \longrightarrow \mathbb{Z}/2\mathbb{Z}=\{0,1\}\}$. $2^{[m]*}$ forms **an algebra** over $\mathbb{Z}/2\mathbb{Z}$ in the usual way, and it has a natural **basis** $\{\delta_a | a \in 2^{[m]}\}$ where δ_a is defined as follows: $\delta_a(b)=1 \Longleftrightarrow b=a$.
- Given a $f \in 2^{[m]*}$, set

$$\text{supp}(f) := f^{-1}(1)$$

- f is said to be **nice** if supp(f) is an abstract simplicial complex.

A one-one correspondence

$$\{\text{all nice functions in } 2^{[m]*}\} \longleftrightarrow \{\text{ all abst. sim. subcpxes in } 2^{[m]}\}.$$

Further development—An algebra-combinatorics formula

Möbius transform

On $2^{[m]*}$, define a $\mathbb{Z}/2\mathbb{Z}$-valued **Möbius transform**

$$\mathcal{M} : 2^{[m]*} \longrightarrow 2^{[m]*}$$

by the following way: for any $f \in 2^{[m]*}$ and $a \in 2^{[m]}$,

$$\mathcal{M}(f)(a) = \sum_{b \subseteq a} f(b)$$

Further development—An algebra-combinatorics formula

The following result indicates an essential relationship between $\mathcal{M}(f)$ and the Betti numbers of $\mathbf{k}(K_f)$.

Algebra–combinatorics formula (Cao-Lü)

Suppose that $f \in 2^{[m]*}$ is nice such that $K_f = \mathrm{supp}(f)$ is an abstract simplicial complex on $[m]$. Then

$$\mathcal{M}(f) = \sum_{i=0}^{h} \sum_{a \in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K_f)} \delta_a$$

where h denotes the length of the minimal free resolution of $\mathbf{k}(K_f)$, and $\beta_{i,a}^{\mathbf{k}(K_f)}$'s denote the Betti numbers of $\mathbf{k}(K_f)$.

An algebra-combinatorics formula

Corollary

$$|\mathrm{supp}(\mathcal{M}(f))| \leq \sum_{i=0}^{h} \sum_{a\in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K_f)}.$$

Proof.
$\mathcal{M}(f) = \sum_{i=0}^{h} \sum_{a\in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K_f)}\delta_a = \sum_{a\in 2^{[m]}} \left(\sum_{i=0}^{h} \beta_{i,a}^{\mathbf{k}(K_f)}\right)\delta_a$
$\Longrightarrow$ for any $a \in \mathrm{supp}(\mathcal{M}(f))$, $\sum_{i=0}^{h} \beta_{i,a}^{\mathbf{k}(K_f)}$ must be odd so $\sum_{i=0}^{h} \beta_{i,a}^{\mathbf{k}(K_f)} \geq 1$. Therefore

$$\sum_{i=0}^{h} \sum_{a\in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K_f)} \geq \sum_{a\in\mathrm{supp}(\mathcal{M}(f))} \sum_{i=0}^{h} \beta_{i,a}^{\mathbf{k}(K_f)} \geq \sum_{a\in\mathrm{supp}(\mathcal{M}(f))} 1 = |\mathrm{supp}(\mathcal{M}(f)$$

Generalized moment-angle complex

Given an abstract simplicial complex K on $[m]$, let $(\underline{X}, \underline{W}) = \{(X_i, W_i)\}_{i=1}^m$ be m pairs of CW-complexes with $W_i \subset X_i$. Then the *generalized moment-angle complex* is defined as follows:

$$K(\underline{X}, \underline{W}) = \bigcup_{\sigma \in K} B_\sigma(\underline{X}, \underline{W}) \subset \prod_{i=1}^{m} X_i$$

where $B_\sigma(\underline{X}, \underline{W}) = \prod_{i=1}^m H_i$ and $H_i = \begin{cases} X_i & \text{if } i \in \sigma \\ W_i & \text{if } i \in [m] \setminus \sigma. \end{cases}$

A class of generalized moment-angle complexes

Take $(\underline{X}, \underline{W}) = (\underline{\mathbb{D}}, \underline{\mathbb{S}}) = \{(\mathbb{D}_i, \mathbb{S}_i)\}_{i=1}^m$ with each CW-complex pair $(\mathbb{D}_i, \mathbb{S}_i)$ subject to the following conditions:

(1) $\mathbb{D}_i$ is acyclic, that is, $\widetilde{H}_j(\mathbb{D}_i) = 0$ for any j.

(2) There exists a unique κ_i such that $\widetilde{H}_{\kappa_i}(\mathbb{S}_i) = \mathbb{Z}$ and $\widetilde{H}_j(\mathbb{S}_i) = 0$ for any $j \neq \kappa_i$.

Then our objective is to calculate the cohomology of

$$\mathcal{Z}_K^{(\underline{\mathbb{D}}, \underline{\mathbb{S}})} := K(\underline{\mathbb{D}}, \underline{\mathbb{S}}) = \bigcup_{\sigma \in K} B_\sigma(\underline{\mathbb{D}}, \underline{\mathbb{S}}) \subset \prod_{i=1}^m \mathbb{D}_i.$$

Further development—Cohomology of a class of generalized moment-angle complexes

Theorem (Cao-Lü)

As graded **k**-modules,

$$H^*(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}) \cong \mathrm{Tor}^{\mathbf{k}[\mathbf{v}]}(\mathbf{k}(K),\mathbf{k}).$$

Corollary

$$\sum_i \dim_{\mathbf{k}} H^i(\mathcal{Z}_K^{(\mathbb{D},\mathbb{S})};\mathbf{k}) = \sum_{i=0}^{h}\sum_{a\in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K)}.$$

Application to Halperin-Carlsson conjecture

Halperin-Carlsson conjecture

If a finite-dimensional paracompact Hausdorff space X admits a free action of a torus T^r (resp. a p-torus $(\mathbb{Z}_p)^r, p$ prime) of rank r, then the total dimension of its cohomology,

$$\sum_i \dim_{\mathbf{k}} H^i(X;\mathbf{k}) \geq 2^r$$

where $\mathbf{k}$ is a field of characteristic 0 (resp. p).

Remark

- Historically, the above conjecture in the p-torus case originates from the work of P. A. Smith in 1950s.
- For the case of a p-torus $(\mathbb{Z}_p)^r$ freely acting on a finite CW-complex homotopic to $(S^n)^k$ suggested by P. E. Conner, the problem has made an essential progress.
- In the general case, the inequality was conjectured by S. Halperin for the torus case, and by G. Carlsson for the p-torus case.
- So far, the conjecture holds if $r \leq 3$ in the torus and 2-torus cases and if $r \leq 2$ in the odd p-torus case. Also, many authors have given contributions to the conjecture in many different aspects.

Lower bound

Recall that

$$\sum_i \dim_{\mathbf{k}} H^i(\mathcal{Z}_{K_f}^{(\mathbb{D},\mathbb{S})};\mathbf{k}) = \sum_{i=0}^{h} \sum_{a\in 2^{[m]}} \beta_{i,a}^{\mathbf{k}(K_f)} \geq |\mathrm{supp}(\mathcal{M}(f))|.$$

We can upbuild a method of compressing $\mathrm{supp}(f)$ to get the desired lower bound of $|\mathrm{supp}(\mathcal{M}(f))|$.

Theorem (Cao-Lü)

For any nice $f \in 2^{[m]*}$, there exists some $a \in \mathrm{supp}(f)$ such that

$$|\mathrm{supp}(\mathcal{M}(f))| \geq 2^{m-|a|}.$$

Application to free actions

Theorem (Cao-Lü)

Let H (resp. $H_{\mathbb{R}}$) be a rank r subtorus of T^m (resp. $(\mathbb{Z}_2)^m$). If H (resp. $H_{\mathbb{R}}$) can act freely on $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$), then

$$\sum_i \dim_{\mathbf{k}} H^i(\mathcal{Z}_K;\mathbf{k}) = \sum_i \dim_{\mathbf{k}} H^i(\mathbb{R}\mathcal{Z}_K;\mathbf{k}) \geq 2^r.$$

Remark

The action of H (resp. $H_{\mathbb{R}}$) on $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$) is naturally regarded as the restriction of the T^m-action Φ to H (resp. the $(\mathbb{Z}_2)^m$-action $\Phi_{\mathbb{R}}$ to $H_{\mathbb{R}}$).

Application to free actions

Corollary

The Halperin–Carlsson conjecture holds for $\mathcal{Z}_K$ (resp. $\mathbb{R}\mathcal{Z}_K$) under the restriction of the T^m-action Φ (resp. the $(\mathbb{Z}_2)^m$-action $\Phi_{\mathbb{R}}$).

Remark

Using a different method, Yury Ustinovsky has also recently proved the Halperin's toral rank conjecture for the moment-angle complexes with the restriction of natural tori actions, see arXiv:0909.1053.