Constructions of three-dimensional small covers

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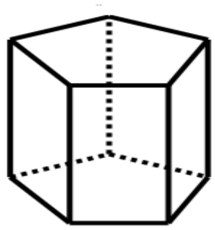
1 Small Cover

Notation

$\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$
$(\mathbb{Z}_2)^n \curvearrowright \mathbb{R}^n \to (\mathbb{R}_{\geq 0})^n$: standard representation
P : simple convex polytope



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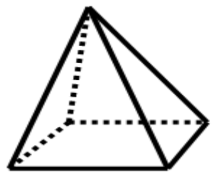

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Definition 1.1 *A small cover over P is an n-dimensional closed manifold M with a locally standard $(\mathbb{Z}_2)^n$-action such that its orbit space is a simple convex polytope P.*

We say that two small covers M_i over P_i $(i = 1, 2)$ are *equivalent* when there exists a θ-equivariantly homeomorphic function $f : M_1 \to M_2$ such that $f(g \cdot x) = \theta(g) \cdot f(x)$ $(g \in (\mathbb{Z}_2)^n,\ x \in M_1)$ for some combinatorially equivalence $\phi : P_1 \to P_2$ and $\theta \in \mathrm{Aut}(\mathbb{Z}_2)^n$.

$$\begin{array}{ccc} M_1 & \xrightarrow{f} & M_2 \\ \downarrow & & \downarrow \\ P_1 & \xrightarrow{\phi} & P_2 \end{array}$$

We say that f is a $\mathrm{GL}(n, \mathbb{Z}_2)$-*equivalence* on ϕ.

Example 1.2 1. $(\mathbb{Z}_2)^n \curvearrowright T^n (= S^1 \times \cdots \times S^1) \to I^n$
2. $(\mathbb{Z}_2)^n \curvearrowright \mathbb{R}P^n \to \Delta^n$

$$(t_1, \cdots, t_n) \cdot [x_0, x_1, \cdots, x_n] = [x_0, (-1)^{t_1} x_1, \cdots, (-1)^{t_n} x_n]$$

$(\mathbb{Z}_2)^n$-colored polytope

$(\mathbb{Z}_2)^n \curvearrowright M \xrightarrow{\pi} P$: small cover
$\mathcal{F}$: facets (codimension 1-faces) of P

Characteristic function
$\lambda : \mathcal{F} \to (\mathbb{Z}_2)^n$

$$\lambda(F) = \text{the generator of the isotropy subgroup at } x \in \pi^{-1}(\text{int}F).$$

$(\star)$ if $F_1 \cap \cdots \cap F_n \neq \emptyset$ then $\{\lambda(F_1), \cdots, \lambda(F_n)\}$ is linearly independent.

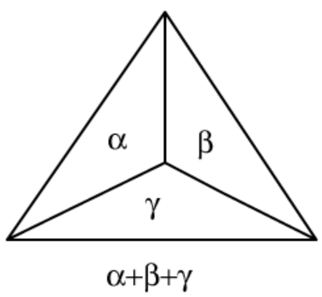
$$M(P,\lambda) := P \times (\mathbb{Z}_2)^n / \sim,$$

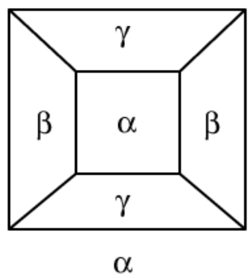
$(x,t) \sim (y,s) \iff x = y \in P,\ s - t \in \langle \lambda(F_1), \cdots, \lambda(F_k) \rangle$ such that $x \in \text{int}(F_1 \cap \cdots \cap F_k)$.

Theorem 1.3 (Davis-Januszkiewicz) *A small cover M over P with a characteristic function λ is* $\mathrm{GL}(n, \mathbb{Z}_2)$*-equivalent (on the identity) to* $M(P,\lambda)$.

We say that two $(\mathbb{Z}_2)^n$-colored polytopes (P_i, λ_i) $(i = 1, 2)$ are *equivalent* when there exist a combinatorially equivalence $\phi : P_1 \to P_2$ such that $\theta \circ \lambda_1 = \lambda_2 \circ \phi$ for some $\theta \in \mathrm{Aut}(\mathbb{Z}_2)^n$.

$(\mathbb{Z}_2)^n$-colored polytopes	$\overset{1:1}{\Longleftrightarrow}$	small covers
(P, λ)	$\longleftrightarrow$	$M(P,\lambda)$





Henceforth we assume that $n = 3$.

Linear model
A small cover with 3-coloring (i.e. $\lambda(\mathcal{F})$ is a basis of $(\mathbb{Z}_2)^3$) is called a *linear model*.

Remark. 3-coloring of P is unique up to equivalence.

P is 3-colorable $\Longleftrightarrow$ each face is an *even* polygon.

Orientable small cover

Theorem 1.4 (N) *A 3-dimensional small cover $M(P, \lambda)$ is orientable if and only if $\lambda(\mathcal{F})$ is contained in $\{\alpha, \beta, \gamma, \alpha+\beta+\gamma\}$ for a suitable basis $\{\alpha, \beta, \gamma\}$ of $(\mathbb{Z}_2)^3$.*

linear model	3-colored polytope	T^3
orientable small cover	4-colored polytope	$\mathbb{R}P^3$
small cover	$(\mathbb{Z}_2)^3$-colored polytope	$S^1 \times \mathbb{R}P^2$

$(\mathbb{Z}_2)^3$**-colorings on** $P^3(3) = I \times \Delta^2$

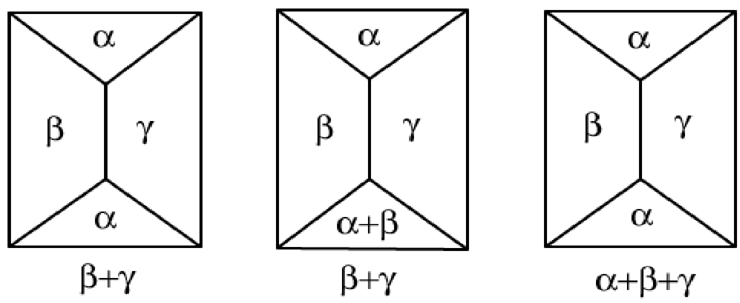
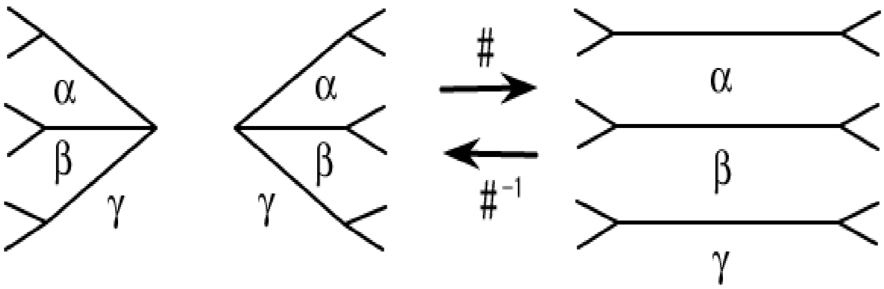


Figure 1: $(\mathbb{Z}_2)^3$-colorings on $P^3(3) = I \times \Delta^2$; λ_1, λ_2 and λ_3

$$M(P^3(3), \lambda_1) \approx M(P^3(3), \lambda_2) \approx S^1 \times \mathbb{R}P^2$$
$$M(P^3(3), \lambda_3) \approx \mathbb{R}P^3 \sharp \mathbb{R}P^3$$

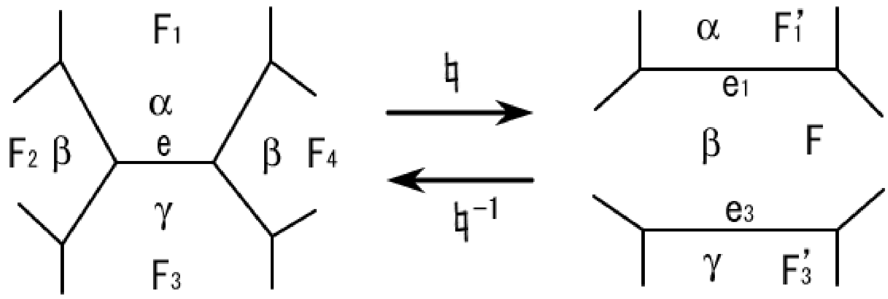
Operations on small covers

Connected sum ♯



$$M((P_1, \lambda_1)\sharp(P_2, \lambda_2)) = M(P_1, \lambda_1)\sharp M(P_2, \lambda_2)$$

Surgery ♮



$$M(\natural(P, \lambda)) = \natural M(P, \lambda), \qquad M(\natural^{-1}(P, \lambda)) = \natural^{-1} M(P, \lambda)$$

The operations $\natural$ and $\natural^{-1}$ both correspond to the ordinaly surgeries on small covers.

Theorem 1.5 (Izmestiev 2001) *Each linear model M^3 can be constructed from T^3 by using the connected sum $\sharp$ and the surgeries $\natural$, $\natural^{-1}$.*

Theorem 1.6 (N 2004 (improved by Kuroki)) *Each orientable small cover M^3 can be constructed from T^3 and $\mathbb{R}P^3$ by using the connected sum $\sharp$ and the surgeries $\natural$, $\natural^{-1}$.*

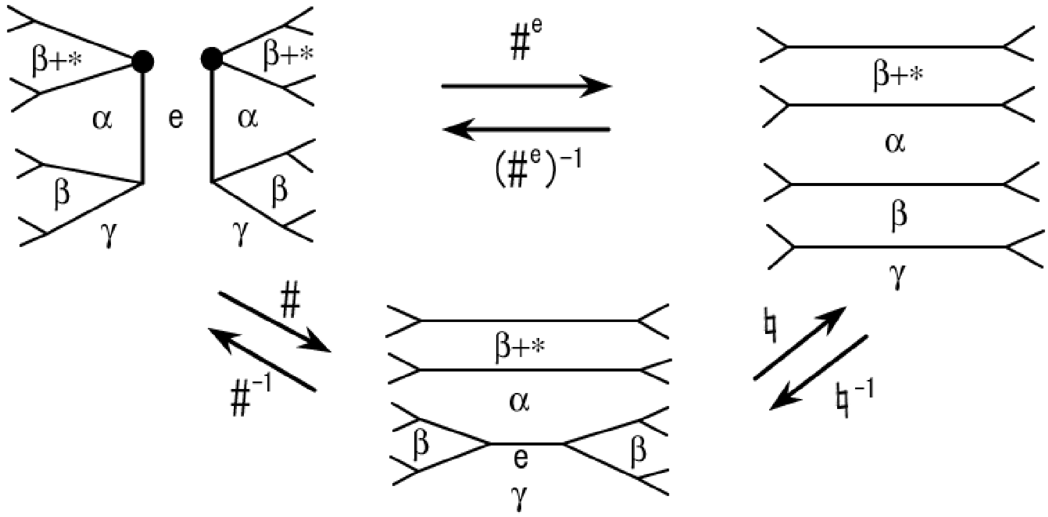
Constructions of general small covers

Problem. What are basic small covers from which we can construct all 3-dimensional small covers using the connected sum $\sharp$ and the surgeries $\natural$, $\natural^{-1}$?

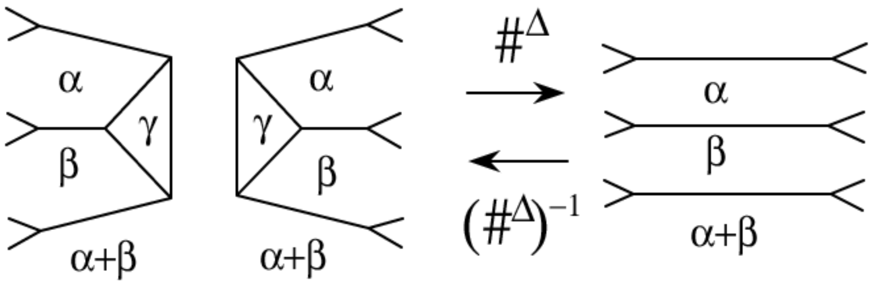
Theorem 1.7 (Lü and Yu 2009) *Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ by using seven operations $\sharp$, $\natural^{-1}$, $\sharp^e$, $\sharp^{eve}$, $\sharp^\Delta$, $\sharp_4^C$ and $\sharp_5^C$.*

Remark (Kuroki). $\sharp^e = \natural \circ \sharp$, $\sharp^{eve} = \natural^2 \circ \sharp$.

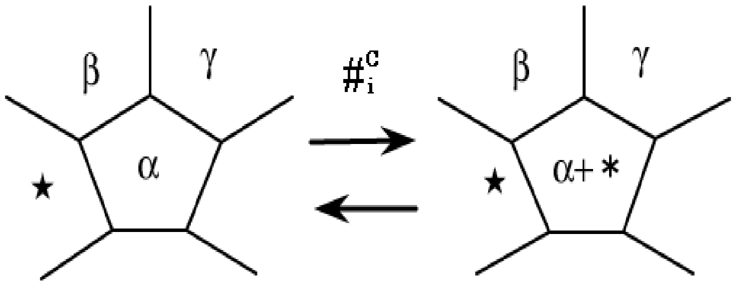
Connected sum along edges $\sharp^e$



Connected sum along 2-independent triangle faces $\sharp^{\Delta}$



Coloring change for a 2-independent i-gon $\sharp_i^C$



Remark. $\sharp_3^C = \sharp^{\Delta}(P^3(3), \lambda_2)$.

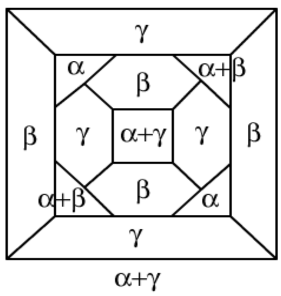
Main theorem

Theorem 1.8 *(1) Each small cover M^3 can be constructed from T^3, $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using the connected sum $\sharp$ and the surgery $\natural$.*
(2) Each small cover M^3 can be constructed from $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with two different $(\mathbb{Z}_2)^3$-actions by using four operations $\sharp$, $\sharp^e$, $\natural^{-1}$ and $\sharp_4^C$.

Corollary 1.9 *(1) Each linear model M^3 can be constructed from T^3 by using two operations $\sharp$ and $\natural$.*
(2) Each orientable small cover M^3 can be constructed from T^3 and $\mathbb{R}P^3$ by using two operations $\sharp$ and $\natural$.

Corollary 1.10 *(1) Each linear model M^3 can be constructed from T^3 by using three operations $\sharp$, $\sharp^e$ and $\natural^{-1}$.*
(2) Each orientable small cover M^3 can be constructed from T^3 and $\mathbb{R}P^3$ by using three operations $\sharp$, $\sharp^e$ and $\natural^{-1}$.

Remark. It is impossible to construct every small covers from basic small covers by using only three operations $\sharp$, $\sharp^e$ and $\natural^{-1}$.



Proof of main theorem.

Definition 1.11 (P,λ) *is quasi-decomposable*
$\overset{def}{\Longleftrightarrow}$ $(P,\lambda)=(P_1,\lambda_1)\sharp(P_2,\lambda_1)$ *or* $(P,\lambda)=(P_1,\lambda_1)\sharp^e(P_2,\lambda_2)$ *except* $P=P_1\sharp^e\Delta^3$.

Proposition 1.12 *Let* (P,λ) *be a* $(\mathbb{Z}_2)^3$*-colored polytope, but not* $P^3(3)$*. If there exist three edges such that they are not adjacent to each other and the* 1*-skeleton of* P *becomes disconnected after cutting them then* (P,λ) *is quasi-decomposable.*

Remark. In this case $(P,\lambda)=(P_1,\lambda_1)\sharp(P_2,\lambda_2)$ or $(P,\lambda)=(P_1,\lambda_1)\sharp^{\Delta}(P_2,\lambda_2)$.

Proposition 1.13 *Let* (P,λ) *be a* $(\mathbb{Z}_2)^3$*-colored polytope. Suppose that the* 3*-connectedness of the* 1*-skeleton of* P *is destroyed after doing surgeries* $\natural$ *or* $\natural^{-1}$*. Then* (P,λ) *is quasi-decomposable.*

Lemma. Each polytope has a face of triangle or quadrilateral or pentagon.

Proposition 1.14 *If* P *is not one of* I^3, $P^3(3)$ *and* Δ^3 *then* P *is quasi-decomposable or* P *can be constructed from some* $(\mathbb{Z}_2)^3$*-colored polytope* Q *such as the following table.*

	3-independent	2-independent	
3-gon	$Q\sharp\Delta^3$	$Q\sharp^e P^3(3)_{hor}$	
4-gon	$Q\sharp^e P^3(3)_{ver}$	$Q\sharp I^3$, $Q\sharp^e I^3$, $\natural(Q\sharp^e I^3)$	$\sharp_4^C(\natural^{-1}Q)$
5-gon	$Q\sharp^e\Delta^3$ ($\to$ 4-gon)	$\natural Q$ ($\to$ 4-gon)	$\natural^{-1}Q$

$(\mathbb{Z}_2)^3$-colorings on I^3

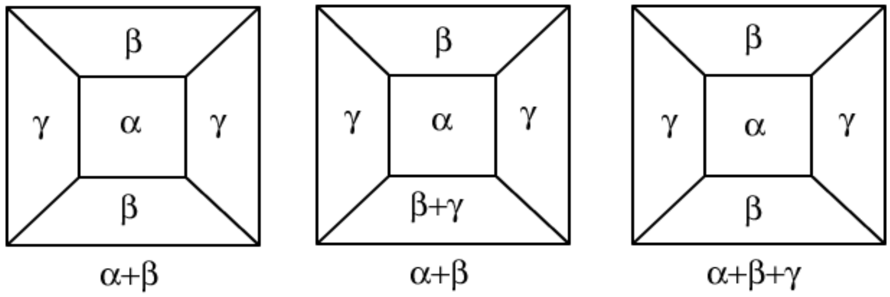


Figure 2: The three $(\mathbb{Z}_2)^3$-coloring on 3-cube I^3; λ_1, λ_2 and λ_3.

Relations of basic $(\mathbb{Z}_2)^3$-colored polytopes.
$(I^3, \lambda_i) = (P^3(3), \lambda_i)\sharp^e(P^3(3), \lambda_i)$ $(i = 1, 2, 3)$
$(P^3(3), \lambda_3) = \Delta^3\sharp\Delta^3$
$(I^3, \lambda_0) = \sharp_4^C(I^3, \lambda_j)$ $(j = 1, 3)$
$(P^3(3), \lambda_1) = (P^3(3), \lambda_2)\sharp^\Delta(P^3(3), \lambda_2)$

Theorem 1.15 *Each $(\mathbb{Z}_2)^3$-colored polytope (P, λ) can be constructed from Δ^3, (I^3, λ_0), $(P^3(3), \lambda_1)$ and $(P^3(3), \lambda_2)$ by using $\sharp$ and $\natural$.*

Theorem 1.16 *Each $(\mathbb{Z}_2)^3$-colored polytope (P, λ) can be constructed from Δ^3, $(P^3(3), \lambda_1)$ and $(P^3(3), \lambda_2)$ by using $\sharp$, $\sharp^e$, $\natural^{-1}$ and $\sharp_4^C$.*