Spherical CR structures on Brieskorn manifold

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- Brieskorn manifold $M(p, q, r)$ is a smooth, compact 3-dimensional manifold obtained by the intersection

$$S^5 \cap \{(z_1, z_2, z_3) \in \mathbb{C}^3 | z_1^p + z_2^q + z_3^r = 0\}.$$

Here p, q, r should be integers ≥ 2.

- This is a typical 3-dimensional Seifert manifold. Brieskorn constructed this manifold related to exotic homotopy sphere.

- The question is: What kind of geometric structure exists on Brieskorn manifold?

- Gromov-Lawson- Thurston '80
  showed the existence of conformally flat structure on $M(p,q,r)$.

  Put $\kappa = p^{-1} + q^{-1} + r^{-1} - 1$. They showed that when $\kappa > 0$ there exist a conformally flat stucture but does not exist when $\kappa = 0$ and exists partially in the case of $\kappa < 0$(depending on p, q, r).

  This result was also shown by Kapovich-Kuiper.

- In this talk we are interested in the case of spherical CR structure. We showed that every $M(p,q,r)$ admits spherical CR structure.

- Spherical CR manifold is a smooth manifold M^{2n+1} modelled on S^{2n+1} whose coordinate changes lie in PU$(n+1,1)$.

- The group PU$(n+1,1)$ is called the unitary Lorentz group which is the group of isometries of the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n+1}$. It acts on $\mathbb{H}_{\mathbb{C}}^{n+1}$ as biholomorphic transformation and extends as Cauchy-Riemann transformation on the boundary sphere S^{2n+1}.

- Therefore, the pair $(\mathsf{PU}(n+1,1), S^{2n+1})$ is called spherical Cauchy-Riemann geometry.

  Let Γ be a subgroup of $\mathsf{PU}(n+1,1)$. If we consider the boundary S^{2n+1} of $\mathbb{H}_{\mathbb{C}}^{n+1}$ and the action of Γ on S^{2n+1}, there is a limit set $L(\Gamma)$ for which Γ acts properly discontinuously on $S^{2n+1} - L(\Gamma)$.

Definition 1. *Let* Γ *be a subgroup of* $\mathrm{PU}(n+1,1)$*. The limit set* $L(\Gamma)$ *is defined to be the set of cluster points of the orbits* $\Gamma.p$ *in* $S^{2n+1}(p \in \mathbb{H}_{\mathbb{C}}^{n+1})$*;*

$$\overline{\Gamma}.p \cap S^{2n+1}$$

- Suppose a manifold M^{2n+1} admits a Spherical CR structure there exist a developing pair

  $$(\rho, \mathrm{dev}) : (\Pi_1(M) = \Pi, \tilde{M}) \longrightarrow (\mathrm{PU}(n+1,1), S^{2n+1})$$

  Proposition 1. *If* $Aut_{CR}(M)$ *is transitive on* M*, then* $f : \tilde{M} \longrightarrow S^{2n+1}$ *is a covering, and* $f(\tilde{M})$ *is a homogeneous domain in* S^{2n+1}

  To state our theorem, we recall Milnor's classification of $M(p,q,r)$.

Let G be a simply connected 3-dimensional Lie group and Π a discrete subgroup of G. Milnor has shown that $M(p,q,r)$ is diffeomorphic to $\Pi\backslash G$. According to the rational number $\kappa = p^{-1}+q^{-1}+r^{-1}-1$, G is as follows:

(i) $\kappa > 0$. $G = \mathrm{SU}(2)$ and Π is a finite subgroup.
(ii) $\kappa = 0$. $G = \mathcal{N}$, the Heisenberg Lie group and Π is a discrete uniform subgroup.
(iii) $\kappa < 0$. $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$, the universal covering of $\mathrm{PSL}(2,\mathbb{R})$ and Π is a cocompact subgroup.

Using this result we state our theorem.

Theorem A. $M(p,q,r)$ *admits a Spherical CR structure which has the "form"* $\Gamma\backslash S^3 - L(\Gamma)$, *where* $\Gamma \subset \mathrm{PU}(2,1)$ *whose holonomy group* Γ *satisfies that:*

(i) $L(\Gamma) = \emptyset \Leftrightarrow \kappa > 0$

(ii) $L(\Gamma) = \{\infty\} \Leftrightarrow \kappa = 0$

(iii) $L(\Gamma) = \mathcal{S}^1 \Leftrightarrow \kappa < 0$

We note the following:

- Limit set $L(\Gamma)$ is not necessarily discrete.
- The above Spherical CR-structure on $M(p,q,r)$ is homogeneous so that dev is a diffeomorphism for $\kappa > 0$, $\kappa = 0$ and an infinite cyclic covering map for $\kappa < 0$.

Proof. To proof our theorem we use one of our theorem

Theorem 1. *Kamishima-Odebiyi*

Let M be a 3-dimensional compact spherical CR manifold. If the developing image misses a point from S^3, then dev : $\tilde{M} \longrightarrow S^3$ *is a covering map onto its image.*

It suffices to construct a developing pair

$$(\rho, \mathsf{dev}) : (\Pi, G) \longrightarrow (\mathsf{PU}(2,1), S^3)$$

$G = \mathsf{SU}(2)$ is identified with S^3 by the orbit map dev$(g) = gx(x = (1,0,0,0) \in S^3)$ .PU$(2,1)$ acts transitively on S^3 and dev is equivariant with respect to the inclusion SU$(2) \to$ PU$(2,1)$ so that $\Pi\backslash G = \Pi\backslash S^3$ such that $gx = S^3 - L(\Pi), \Gamma = \rho(\Pi_1(M)) = \Pi$. In this case dev is a diffeomorphism .

To show that $L(\Pi) = \emptyset$. Recall that $L(\Pi) = \overline{\Pi}.p \cap S^3$. If Π is infinite, $L(\Pi)$ is not empty. If Π is finite and compact, $L(\Pi)$ has a fixed point in $\mathbb{H}_{\mathbb{C}}^{n+1}$ so that $L(\Pi) = \emptyset$.

When $\kappa = 0$, the Heisenberg group $\mathcal{N}$ is a 2-step nilpotent Lie group such that $C^2\mathcal{N} = \mathbb{R}$ .It is defined by the group law $(\zeta, v) * (\xi, w) \mapsto (\zeta + \xi, v + w + 2I(\zeta\bar{\xi}))$ for $\zeta, \xi \in \mathbb{C}$ and $v, w \in \mathbb{R}$. Since the boundary of $\mathbb{H}_{\mathbb{C}}^{n+1}$ can be identified with the one point comactification of $\mathcal{N}$, $\mathcal{N} = S^3 - \{\infty\}$. It suffices to construct a developing pair

$$(\rho, \mathsf{dev}) : (\Pi, G) \longrightarrow (\mathsf{PU}(2,1), S^3 - \{\infty\})$$

The Heisenberg group is a subgroup of $\mathsf{PU}(2,1)$ and it acts simply transitively on $S^3 - \{\infty\}$ so that $\Pi\backslash G = \Pi\backslash\mathcal{S}^3 - \{\infty\}$. In this case dev is a diffeomorphism . To show that $L(\Pi) = \{\infty\}$, note that for $\Pi = \rho(\Pi)$ in $\mathsf{PU}(2,1)$, Π fixes $\{\infty\}$ such that $L(\Pi) = \{\infty\}$

In the case when $\kappa < 0$. Recall that $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathrm{PO}(2,1)^0$ or $\mathrm{PU}(1,1)$. Suppose $\varphi\colon \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{PO}(2,1)^0$ is an isomorphism. Let $\tilde{\varphi}\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \widetilde{\mathrm{PO}(2,1)^0}$ be the isomorphism of the universal covering groups. As $\widetilde{\mathrm{PO}(2,1)^0}$ acts transitively on $\widetilde{S^3 - S^1} = \widetilde{T_1\mathbb{H}^2_{\mathbb{R}}}$, , we have a diffeomorphism:

$$\widetilde{\mathrm{dev}}\colon G = \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \widetilde{S^3 - S^1} \tag{1}$$

such that $\widetilde{\mathrm{dev}}(g) = \tilde{\varphi}(g)\cdot\tilde{x}(\tilde{x} \in \widetilde{S^3 - S^1})$. Since $\widetilde{\mathrm{dev}}$ is equivariant with respect to $\tilde{\varphi}$ we have $\Pi\backslash G \cong \tilde{\varphi}(\Pi)\backslash\widetilde{S^3 - S^1}$ which admits a developing map as the projection :

$$(\rho, \mathsf{dev})\colon (\Pi, G) \to (\mathsf{PO}(2,1), T_1\mathbb{H}^2_{\mathbb{R}})$$

such that the limit set $L(\rho(\Pi)) = S^1 = \partial\mathbb{H}^2_{\mathbb{R}}$.

$\square$

- In other words, because of the existence of transitive action on $\widetilde{M}(p,q,r)$ we get a homogeneous developing pair, which implies dev is a covering map onto its image.

- Having got spherical CR structure on $M(p,q,r)$ whose developing pair is homogeneous, we ask the question:

  Problem 1. *When $\kappa < 0$,does there exist any spherical CR structure on $M(p,q,r)$ whose developing pair is not homogeneous?*

- For the homogeneous spherical CR structure,when $\kappa < 0$, $L(\Gamma)$ is a geometric circle.

- To get a non homogeneous spherical structure on $M(p, q, r)$, since $L(\Gamma)$ would not be a geometric circle then it should be a non rectifiable Jordan curve.

  To construct such a spherical CR structure on $M(p, q, r)$, we would try to explain an example of such case where $L(\Gamma)$ is a non rectifiable curve from the result of Goldman-Kapovich-Leeb '01.

If we take a compact real hyperbolic surface $\mathbb{H}^2_{\mathbb{R}}/\Gamma_1$ and a compact complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}/\Gamma_2$. Sewing along the common geodesic circle $S^1 = \mathbb{H}^1_{\mathbb{R}}/\mathbb{Z}$ gives a closed surface Σ.

Toledo considered an invariant associated to a representation $\rho : \pi_1(\Sigma) \to \mathrm{PU}(2,1)$. This representation determines a flat bundle . $D : \mathbb{H}^2_{\mathbb{C}}/\Gamma \to \Sigma$ This bundle has a section that is equivalent to an equivariant mapping $\tilde{f} : \tilde{\Sigma} \to \mathbb{H}^2_{\mathbb{C}}$, where $\tilde{\Sigma}$ is the universal cover of Σ.The integral of the pull back $\tilde{f}^*\omega$ of the kaehler form ω on $\mathbb{H}^2_{\mathbb{C}}$ is the characteristic number

$$\tau(\rho : \tilde{f}) := \frac{1}{2\pi} \int_{\Sigma} f^*\omega$$

called the Toledo invariant τ.

We state the result by Golman-Kapovich-Leeb.

Theorem 2. *For every genus $g \geq 2$ and every even integer τ satisfying*

$$2 - 2g \leq \tau \leq 2g - 2 \tag{2}$$

there exists a convex-cocompact discrete and faithful reperesentation $\rho : \pi_1(\Sigma) = \pi \to \mathrm{PU}(2,1)$ with $\tau(\rho) = \tau$. Furthermore, the complex hyperbolic surface $M = \mathbb{H}^2_{\mathbb{C}}/(\rho(\pi)) = \Gamma$ is diffeomorphic to the total space of an oriented $\mathbb{R}^2$-bundle ξ over Σ with the Euler number

$$e(\xi) = \chi(\Sigma) + \mid \tau(\rho)/2 \mid \tag{3}$$

From theorem 2 we see that there is an oriented disk bundle over Σ . We recall from the above result that the convex core is taken such that for the smallest convex submanifold the inclusion into the whole space is homotopy equivalent to the whole space. This implies for the map $\Pi^{-1} : \Sigma \to \mathbb{H}^2_{\mathbb{C}}/\Gamma$ by pushing forward $\partial\Pi^{-1}(\Sigma)$ to the boundary , $\partial\Pi^{-1}(\Sigma) \cong S^3 - L(\Gamma)$. Note that $L(\Gamma)$ is a non rectifiable curve.

Therefore as an extension of theorem 2 we have the bundle

$$\begin{array}{ccccccc}
\partial D^2 & \longrightarrow & \partial(\Pi^{-1}(\Sigma)))/\Gamma & \longrightarrow & \Pi^{-1}(\Sigma) & \xrightarrow{p} & \Sigma \\
|| & & || & & & & || \\
S^1 & \longrightarrow & S^3 - L(\Gamma)/\Gamma & \longrightarrow & & \longrightarrow & \Sigma
\end{array}$$

Brieskorn manifold continued.

We would like to list the possibilities of $\kappa = p^{-1} + q^{-1} + r^{-1} - 1$. Note that (p, q, r) is listed such that $p \leq q \leq r$.

- When $\kappa > 0$,(p,q,r) must be one of the triples
$$(2, 3, 3), (2, 3, 4), (2, 3, 5) or (2, 2, r)$$
for some $r \geq 2$.

- When $\kappa = 0$, the triple (p, q, r) must be either
$$(2, 3, 6), (2, 4, 4)\text{or}(3, 3, 3).$$

- In the last case when $\kappa < 0$ we have the infinitely remaining triples.

We are interested in the case when $\kappa < 0$.

We recall the following results by Milnor:

Lemma 1. *The Brieskorn manifold $M(p,q,r)$ has a finite covering manifold diffeomorphic to a non-trivial circle bundle over a surface.*

We have the bundle map

$$\begin{array}{ccccc} & & S^1 & = & S^1 \\ & & \downarrow & & \downarrow \\ finite & \longrightarrow & L(p,q,r) & \xrightarrow{\tilde{f}} & M(p,q,r) \\ & & \downarrow & & \downarrow \\ finite & \longrightarrow & \Sigma^* & \xrightarrow{f} & \Sigma_g \end{array} \tag{4}$$

Considering the bundle map in the previous slide, we know that the Euler number of a bundle belongs to the cohomology group of its base space which means $e(M(p,q,r)) \in H^2(\Sigma_g)$ and $e(L(p,q,r)) \in H^2(\Sigma^*)$. $e(M(p,q,r)) \in H^2(\Sigma_g : \mathbb{Z}) \cong \mathbb{Z}$, $H_2(\Sigma_g : \mathbb{Z}) = \mathbb{Z} =< [\Sigma_g] >$. Therefore $< e(M(p,q,r)), [\Sigma_g] >= n \in Z$.

Similarly,

$$\begin{aligned} < e(L(p,q,r)), [\Sigma^*] > &=< f^* e(M(p,q,r)), [\Sigma^*] > \\ &=< e(M(p,q,r)), f^*[\Sigma^*] > \\ &=< e(M(p,q,r)), l[\Sigma_g] > \\ &=< le(M(p,q,r)), [\Sigma_g] > \\ &= ln \end{aligned} \tag{5}$$

Theorem 3. *If the least common multiple* m, *of* (p,q),(p,r) *and of* (q,r) *are all equal:*

$$m = l.c.m(p,q) = l.c.m(p,r) = l.c.m(q,r), \quad (6)$$

then the Brieskorn manifold $M(p,q,r)$ *fibers as a smooth circle bundle with chern number* $-pqr/m^2$ *over a Riemann surface of Euler charactersitic* $pqr(p^{-1}+q^{-1}+r^{-1})/m$ *.*

- $M(p,q,r)$ is a principal circle bundle over a Riemann surface. The free action is defined as follows for $t \in S^1$:

$$t(z_1,z_2,z_3) = (t^{\frac{m}{p}}z_1, t^{\frac{m}{q}}z_2, t^{\frac{m}{r}}z_3)$$

- We have shown an example from the result of Goldman-Kapovich-Leeb of an S^1 bundle $S^3 - L(\Gamma)/\Gamma$ over Σ_g.

- On the other hand, when $\kappa < 0$ $M(p, q, r)$ is a principal circle bundle over an oriented closed surface.

  We state the remark below

Remark 1. *When $\kappa < 0$, a finite cover $L(p,q,r)$ of $M(p,q,r)$ is a nontrivial circle bundle over an oriented closed surface. Take a compact real hyperbolic surface $\mathbb{H}^2_{\mathbb{R}}/\Gamma_1$ and a compact complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}/\Gamma_2$. Sewing along the common geodesic circle $S^1 = \mathbb{H}^1_{\mathbb{R}}/\mathbb{Z}$ gives a closed surface Σ. It has been shown by Goldman-kapovich-Leeb that there is a faithful representation of $\pi_1(\Sigma) = \Gamma$ into* $\mathrm{PU}(2,1)$ *whose limit set $L(\Gamma)$ is a topological (non-rectifiable) circle. Moreover $S^3 - L(\Gamma)/\Gamma$ is a* nontrivial *circle bundle over Σ. The finite cover $L(p,q,r)$ of $M(p,q,r)$ is diffeomorphic to $S^3 - L(\Gamma)/\Gamma$.*

Proof. Two S^1 bundles are equivalent if and only if their Euler numbers are equal. Hence we would like to show that $e(L(p,q,r)) = e(S^3 - L(\Gamma)/\Gamma)$.

First, we substitute $e(M(p,q,r))$ and $\chi(\Sigma)$ of theorem 3 into equation (3) of theorem 2. We have

$$-pqr/m^2 = pqr(p^{-1} + q^{-1} + r^{-1} - 1)/m + \mid \tau(\rho)/2 \mid \quad (7)$$

The next step is to find a Toledo invariant τ satisfying the above equation.

To find such a τ we have to substitute the possible values of the triples (p, q, r) satisfying the condition (6) $m = l.c.m(p, q) = l.c.m(p, r) = l.c.m(q, r)$ to calculate $\mid \tau(\rho)/2 \mid$ satisfying condition (2) $2 - 2g \leq \tau \leq 2g - 2$. We recall that in the case when $\kappa < 0$, the triples (p, q, r) are infinitely many.

Clearly, when $p = q = r$ for $p, q, r > 3$ the condition (6) is satisfied.Other possibilities include

	(p,q,r)		(p,q,r)		(p,q,r)	
	$\pm\tau$	g	$\pm\tau$	g	$\pm\tau$	g
p=q=r	(4,4,4)		(5,5,5)		(6,6,6)	
	0	3	10	6	24	10
p = 2	(2,5,10)		(2,7,14)		(2,9,18)	
	2	2	6	3	10	4
p= 3	(3,4,12)		(3,5,15)		(3,7,21)	
	6	3	10	4	18	6
p = 4	(4,5,20)		(4,6,24)		(4,7,28)	
	18	6	24	7.5	30	9
p = 5	(5,6,30)		(5,7,35)		(5,8,40)	
	34	10	42	12	50	14

p = 6	(6,7,42)		(6,8,48)		(6,9,54)	
	54	15	64	17.5	74	20
p = 7	(7,8,56)		(7,9,63)		(7,10,70)	
	78	21	90	24	102	27
p = 8	(8,9,72)		(8,10,80)		(8,11,88)	
	106	28	120	31.5	134	35
p = 9	(9,10,90)		(9,11,99)		(9,12,108)	
	138	36	154	40	170	44
p= 10	(10,11,110)		(10,12,120)		(10,13,130)	
	174	45	192	49.5	210	54

We see that except for the case when $(p,q,r) = (4,4,4)$ all the possibilities satisfying condition (6) satisfies condition (2).

Hence $(e(L(p,q,r)) = e(S^3 - L(\Gamma)/\Gamma)$ which implies the non trivial circle bundle $L(p,q,r)$ is diffeomorphic to the non trivial circle bundle $S^3 - L(\Gamma)/\Gamma$. $\square$

In summary,

- We have shown that in all cases of κ, $M(p,q,r)$ admits a homogeneous spherical CR structure.

- The nontrivial circle bundle $S^3 - L(\Gamma)$ over Σ_g

  Example of the case when $L(\Gamma)$ is topological circle.

- The bundle in this case is diffeomorphic to $M(p,q,r)$ when $\kappa < 0$.

	(p,q,r)		(p,q,r)		(p,q,r)		$(p,q,$	
	τ	g	τ	g	τ	g	τ	
p=q=r	(4,4,4)		(5,5,5)		(6,6,6)		(7,7,	
	0	0	10	6	24	13	42	
p = 2	(2,5,10)		(2,7,14)		(2,9,18)		(2,11,	
	2	2	6	4	10	6	14	
p= 3	(3,4,12)		(3,5,15)		(3,7,21)		(3,8,2	
	6	4	10	6	18	10	22	
p = 4	(4,5,20)		(4,6,24)		(4,7,28)		(4,9,3	
	18	10	24	13	30	16	42	
p = 5	(5,6,30)		(5,7,35)		(5,8,40)		(5,9,4	
	34	18	44	22	50	26	58	
p = 6	(6,7,42)		(6,8,48)		(6,9,54)		(6,10,	
	54	28	64	33	74	38	84	
p = 7	(7,8,56)		(7,9,63)		(7,10,70)		(7,11,	
	78	40	90	46	102	52	114	
p = 8	(8,9,72)		(8,10,80)		(8,11,88)		(8,12,	
	106	54	120	61	134	68	148	
p = 9	(9,10,90)		(9,11,99)		(9,12,108)		(9,13,1	
	138	70	154	78	170	86	186	