Spherical CR structure on Brieskorn manifold

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Outline of my talk

1. Introduction and definitions
2. Statement of theorem and proof
3. Example: Goldman-Kapovich-Leeb
4. Example: J. Milnor
5. Theorem and proof
6. Summary and conclusion

Brieskorn Manifold

Definition

Brieskorn manifold $M(p,q,r)$ is a smooth, compact 3-dimensional manifold obtained by the intersection

$$S^5 \cap \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^p + z_2^q + z_3^r = 0\}.$$

p, q, r are integers ≥ 2.

- This is a typical 3-dimensional Seifert manifold.
- What kind of geometric structure exists on Brieskorn manifold?

Conformally flat structure on Brieskorn manifold.

Gromov-Lawson-Thurston '88 showed the existence of conformally flat structure on $M(p, q, r)$.
Put $\kappa = p^{-1} + q^{-1} + r^{-1} - 1$. They showed that when $\kappa > 0$ there exist a conformally flat stucture but does not exist when $\kappa = 0$ and it exists partially in the case of $\kappa < 0$(depending on p, q, r).
This result was also shown independently by Kapovich and Kuiper.

In this talk we are interested in the case of spherical CR structure. We showed that every $M(p,q,r)$ admits spherical CR structure.

Definition

Spherical CR manifold is a smooth manifold M^{2n+1} modelled on S^{2n+1} whose coordinate changes lie in $\mathrm{PU}(n+1,1)$.

The group $\mathrm{PU}(n+1,1)$ is called the unitary Lorentz group which is the group of isometries of the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^{n+1}$. It acts on $\mathbb{H}_{\mathbb{C}}^{n+1}$ as biholomorphic transformation and extends as Cauchy-Riemann transformation on the boundary sphere S^{2n+1}.

Spherical CR structure cont'd

Therefore, the pair $(\mathrm{PU}(n+1,1), S^{2n+1})$ is called spherical Cauchy-Riemann geometry.

- Suppose a manifold M^{2n+1} admits a spherical CR structure there exist a developing pair

$$(\rho, \mathrm{dev}) : (\Pi_1(M) = \Pi, \tilde{M}) \longrightarrow (\mathrm{PU}(n+1,1), S^{2n+1}).$$

such that $\mathrm{dev} \circ \gamma = \rho(\gamma) \circ \mathrm{dev} \quad (\forall \gamma \in \Pi)$. Where ρ is holonomy homomorphism and dev is the developing map.

Limit set

Let Γ be a subgroup of $\mathrm{PU}(n+1,1)$. If we consider the boundary S^{2n+1} of $\mathbb{H}_{\mathbb{C}}^{n+1}$ and the action of Γ on S^{2n+1}, there is a limit set $L(\Gamma)$ for which Γ acts properly discontinuously on the domain $S^{2n+1} - L(\Gamma)$.

Definition

Let Γ be a subgroup of $\mathrm{PU}(n+1,1)$. The limit set $L(\Gamma)$ is defined to be the set of cluster points of the orbits $\Gamma \cdot p$ in $S^{2n+1} (p \in \mathbb{H}_{\mathbb{C}}^{n+1})$.

Milnor's classification of $M(p,q,r)$

To state our theorem, we recall Milnor's classification of $M(p,q,r)$. Let G be a simply connected 3-dimensional Lie group and Π a discrete subgroup of G. Milnor has shown that $M(p,q,r)$ is diffeomorphic to $\Pi\backslash G$. According to the rational number $\kappa = p^{-1} + q^{-1} + r^{-1} - 1$, G is as follows:

1. $\kappa > 0. G = \mathrm{SU}(2)$ and Π is a finite subgroup.
2. $\kappa = 0. G = \mathcal{N}$, the Heisenberg Lie group and Π is a discrete uniform subgroup.
3. $\kappa < 0. G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$, the universal covering of $\mathrm{PSL}(2,\mathbb{R})$ and Π is a cocompact subgroup.

Using this result we state our theorem.

Theorem

Theorem (Ka-Od)

$M(p,q,r)$ admits a Spherical CR structure which has the "form" $\Gamma\backslash S^3 - L(\Gamma)$, where the holonomy group $\Gamma \subset \mathrm{PU}(2,1)$ satisfies that:

- $L(\Gamma) = \emptyset \Longleftrightarrow \kappa > 0$
- $L(\Gamma) = \{\infty\} \Leftrightarrow \kappa = 0$
- $L(\Gamma) = \mathcal{S}^1 \Leftrightarrow \kappa < 0$

We note the following:

- Γ is not necessarily discrete.
- The above Spherical CR-structure on $M(p,q,r)$ is homogeneous so that dev is a diffeomorphism for $\kappa > 0, \kappa = 0$ and an infinite cyclic covering map for $\kappa < 0$.

Preamble to proof

We use the following results to proof our theorem.

Proposition

Suppose M admits a transitive CR group then the developing map is a covering onto its image which is a homogeneous domain.

Using this result we give a CR structure on $M(p, q, r)$.

Proof of theorem

When $\kappa < 0$, it suffices to construct a developing pair

$$(\rho, \mathrm{dev}) : (\Pi, G) \longrightarrow (\mathrm{PU}(2,1), S^3)$$

$\mathrm{PU}(2,1)$ acts transitively on S^3. $\mathrm{SU}(2) \subset \mathrm{PU}(2,1)$ such that $G = \mathrm{SU}(2)$ is identified with S^3 by the orbit map $\mathrm{dev}(g) = gx \, (x = (1,0,0,0) \in S^3)$. Obviously, dev is a diffeomorphism. Therefore $\Pi \backslash G = \Pi \backslash S^3$.
We note that $L(\Pi) = \emptyset$ because Π is finite.

Proof of theorem

When $\kappa = 0$,the Heisenberg group $\mathcal{N}$ is a 2-step nilpotent Lie group i.e $C^2(\mathcal{N}) = 0$ .It is defined as a group by the group law $(\zeta, v) * (\xi, w) \mapsto (\zeta + \xi, v + w + 2Im(\zeta\bar{\xi}))$ for $\zeta, \xi \in \mathbb{C}^n \quad v, w \in \mathbb{R}$ and as a manifold by $\mathbb{C}^n \times \mathbb{R}$. Since the boundary of $\mathbb{H}^{n+1}_{\mathbb{C}}$ can be identified with the one point compactification of $\mathcal{N}$, $\mathcal{N} = S^3 - \{\infty\}$. It suffices to construct a developing pair

$$(\rho, \mathrm{dev}) : (\Pi, G) \longrightarrow (\mathrm{PU}(2,1), S^3 - \{\infty\})$$

The Heisenberg group is a subgroup of $\mathrm{PU}(2,1)$ and it acts simply transitively on $S^3 - \{\infty\}$. dev is a diffeomorphism so that $\Pi \backslash G = \Pi \backslash S^3 - \{\infty\}$. Π fixes $\{\infty\}$ such that $L(\Pi) = \{\infty\}$

Proof of theorem

In the last case when $\kappa < 0$. Recall that $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathrm{PO}(2,1)^0$ or $\mathrm{PU}(1,1)$. Suppose $\varphi\colon \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{PO}(2,1)^0$ is an isomorphism. Let $\tilde{\varphi}\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \widetilde{\mathrm{PO}(2,1)^0}$ be the isomorphism of the universal covering groups. As $\widetilde{\mathrm{PO}(2,1)^0}$ acts transitively on $\widetilde{S^3 - S^1} = \widetilde{T_1\mathbb{H}^2_{\mathbb{R}}}$, we have a diffeomorphism:

$$\widetilde{\mathrm{dev}}\colon G = \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \widetilde{S^3 - S^1}$$

such that $\widetilde{\mathrm{dev}}(g) = \tilde{\varphi}(g) \cdot \tilde{x} (\tilde{x} \in \widetilde{S^3 - S^1})$.

Proof of theorem

Since $\widetilde{\mathrm{dev}}$ is equivariant with respect to $\tilde{\varphi}$ we have $\Pi \backslash G \cong \tilde{\varphi}(\Pi)\backslash \widetilde{S^3 - S^1}$ which admits a developing map as the projection :

$$(\rho, \mathrm{dev})\colon (\Pi, G) \to (\mathrm{PO}(2,1), T_1\mathbb{H}^2_{\mathbb{R}})$$

such that the limit set $L(\rho(\Pi)) = S^1 = \partial\mathbb{H}^2_{\mathbb{R}}$.

Summary of proof

In other words, because of the existence of transitive action on $\widetilde{M}(p, q, r)$ we get a homogeneous developing pair, which implies dev is a covering map onto its image.
Having got spherical CR structure on $M(p, q, r)$ such that the image of the developing map is homogeneous, we ask the question:

Problem

When $\kappa < 0$,does there exist any spherical CR structure on $M(p, q, r)$ such that the image of developing map is not homogeneous?

To consider this problem we showed the theorem below

Theorem (Ka-Od)

Let M be a 3-dimensional compact spherical CR manifold. If the developing image misses a point from S^3, then $\mathrm{dev} : \tilde{M} \longrightarrow S^3$ is a covering map onto its image.

In relation to the above theorem, when $\kappa < 0$,the image is $S^3 - S^1$. If the developing image is non homogeneous and $L(\Gamma)$ is not a geometric circle but still connected then $L(\Gamma)$ might be a non rectifiable curve(Topological circle).

Example G-K-L

In second half of this talk we give an example of a spherical CR structure that is non homogeneous such that $L(\Gamma)$ is a non rectifiable curve from the result of Goldman-Kapovich-Leeb(G-K-L).

If we take a compact real hyperbolic surface $\mathbb{H}^2_{\mathbb{R}}/\Gamma_1$ and a compact complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}/\Gamma_2$. Sewing along the common geodesic circle $S^1 = \mathbb{H}^1_{\mathbb{R}}/\mathbb{Z}$ gives a closed surface Σ.

G-K-L considered a representation $\rho : \pi_1(\Sigma) \to \mathrm{PU}(2,1)$.

Example G-K-L

Theorem (Goldman-Kapovich-Leeb)

For every genus $g \geq 2$ *and every even integer* τ *satisfying*

$$2 - 2g \leq \tau \leq 2g - 2$$

there exists a convex-cocompact discrete and faithful representation $\rho : \pi_1(\Sigma) = \pi \rightarrow \mathrm{PU}(2,1)$, *(*$\rho(\pi) = \Gamma$*)with* $\tau(\rho) = \tau$*. Furthermore, the complex hyperbolic surface* $M = \mathbb{H}^2_{\mathbb{C}}/\Gamma$ *is diffeomorphic to the total space of an oriented* $\mathbb{R}^2$*-bundle* ξ *over* Σ *with the Euler number*

$$e(\xi) = \chi(\Sigma) + \mid \tau(\rho)/2 \mid$$

τ is the Toledo invariant.

Example G-K-L

If we take the convex core $C(\mathbb{H}^2_{\mathbb{C}}/\Gamma)$ of $\mathbb{H}^2_{\mathbb{C}}/\Gamma$ by pushing forward the boundary $\partial C(\mathbb{H}^2_{\mathbb{C}}/\Gamma)$ we obtain $S^3 - L(\Gamma)/\Gamma$.
Note that $L(\Gamma)$ is a topological circle.

Corollary

The manifold $S^3 - L(\Gamma)/\Gamma$ is diffeomorphic to the total space of an S^1- bundle over the surface Σ which has the same Euler number as the $\mathbb{R}^2$ fibration of $M = \mathbb{H}^2_{\mathbb{C}}/\Gamma$.

We would like to list the possibilities of $\kappa = p^{-1} + q^{-1} + r^{-1} - 1 (p \leq q \leq r)$.

- When $\kappa > 0$,(p, q, r) must be one of the triples

$$(2, 3, 3), (2, 3, 4), (2, 3, 5) or (2, 2, r)$$

  for some $r \geq 2$.

- When $\kappa = 0$, the triple (p, q, r) must be either

$$(2, 3, 6), (2, 4, 4) \text{or} (3, 3, 3).$$

- In the last case when $\kappa < 0$ we have the infinitely remaining triples.

We are interested in the case when $\kappa < 0$.

Example J.M

Theorem (J. Milnor)

If the least common multiples of (p, q),(p, r) and of (q, r) are all equal

$$m = l.c.m(p, q) = l.c.m(p, r) = l.c.m(q, r),$$

then the Brieskorn manifold $M(p, q, r)$ fibers as a smooth circle bundle with chern number $-pqr/m^2$ over a Riemann surface of Euler charactersitic $pqr(p^{-1} + q^{-1} + r^{-1} - 1)/m$.

$M(p, q, r)$ is a principal circle bundle over a Riemann surface. The free action is defined as follows for $t \in S^1$:

$$t(z_1, z_2, z_3) = (t^{\frac{m}{p}} z_1, t^{\frac{m}{q}} z_2, t^{\frac{m}{r}} z_3)$$

We have two S^1 bundle from the two results. Two S^1 bundles over the same surface are equivalent if and only if their Euler numbers are equal.Hence we would like to show that $e(M(p,q,r)) = e(S^3 - L(\Gamma)/\Gamma)$ if we take same Σ.
First, we substitute $e(M(p,q,r))$ and $\chi(\Sigma)$ into the equation

$$e(\xi) = \chi(\Sigma) + \mid \tau(\rho)/2 \mid$$

We have

$$-pqr/m^2 = pqr(p^{-1} + q^{-1} + r^{-1} - 1)/m + \mid \tau(\rho)/2 \mid$$

The next step is to find a Toledo invariant τ satisfying the above equation.

To find such a τ we have to substitute the possible values of the triples (p,q,r) satisfying the condition

$$m = l.c.m(p,q) = l.c.m(p,r) = l.c.m(q,r)$$

to calculate $\mid \tau(\rho)/2 \mid$ satisfying condition that for every $g \geq 2$

$$2 - 2g \leq \tau \leq 2g - 2$$

Clearly, when $p = q = r$ for $p, q, r > 3$ the first condition is satisfied.

Theorem

Theorem

When $p = q = r$ *for* $p, q, r > 3$*, the difference* $\mid \tau(\rho)/2 \mid = e(M(p,q,r)) - \chi(\Sigma)$ *is* $m^2 - 4m$ *such that* $\mid \tau \mid = 2m^2 - 8m$ *is an even integer with* $g \geq 2$ *satisfying*

$$2 - 2g \leq \tau \leq 2g - 2.$$

Example

For $M(5,5,5), m = 5$, note that $e = -m$ and
$\chi = 3m - m^2 = 2 - 2g$. Therefore, $\tau = \pm 10$ and $g = 6$.

Other possibilities

We recall that in the case when $\kappa < 0$, the triples (p, q, r) are infinitely many.

Other possibilities and their values are:

We see that so far all the possibilities satisfying first condition satisfies second.

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Hence $(e(M(p,q,r)) = e(S^3 - L(\Gamma)/\Gamma)$ which implies that the non trivial circle bundle $M(p,q,r)$ is diffeomorphic to the non trivial circle bundle $S^3 - L(\Gamma)/\Gamma$.
Note that $S^3 - L(\Gamma)/\Gamma$ is non homogeneous. If it were homogeneous then $e(S^3 - L(\Gamma)/\Gamma) = \chi(\Sigma)$ for the representation $\Gamma_1 \hookrightarrow \mathrm{PO}(2,1)$ and $e(S^3 - L(\Gamma)/\Gamma) = \frac{1}{2}\chi(\Sigma)$ for the representation $\Gamma_2 \hookrightarrow \mathrm{PU}(1,1)$. But $e(S^3 - L(\Gamma)/\Gamma)$ is only equal to $\chi(\Sigma)$ in the case $M(4,4,4)$ because $\tau = 0$ where $g = 3$.

Therefore, When $p = q = r$ for $p, q, r > 4$, $M(p, q, r)$ admits non homogeneous spherical CR structure and $L(\Gamma)$ must be a non rectifiable(Topological) circle.