Properties of Bott Manifolds and Cohomological Rigidity

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1. Bott tower and Bott manifold

Bott Tower

$$B_n \to B_{n-1} \to \cdots \to B_i \to B_{i-1} \to \cdots \to B_1 \to B_0$$

$$B_i = P(\xi_{i1} \oplus \xi_{i2}), \qquad B_1 = \mathbb{C}P^1 = P(\mathbb{C} \oplus \mathbb{C}), \qquad B_0 = \{pt\}$$

ξ_{i1}, ξ_{i2}: $\mathbb{C}$- line bundles over B_{i-1}

Each B_i is called an (i-th stage) Bott manifold

In general

$$P(\eta_1 \oplus \eta_2) \underset{\text{difeom}}{\cong} P(\underset{\text{any } \mathbb{C}\text{-line bundle}}{\xi} \otimes (\eta_1 \oplus \eta_2))$$

$$\therefore \quad P(\eta_1 \oplus \eta_2) \cong P(\bar{\eta}_1 \otimes (\eta_1 \oplus \eta_2))$$

$$\cong P(\underline{\mathbb{C}} \oplus (\bar{\eta}_1 \otimes \eta_2))$$

Hence we may consider $B_i = P(\underset{\text{trivial bundle}}{\underline{\mathbb{C}}} \oplus \underset{\mathbb{C}\text{-line bundle}/B_{i-1}}{\xi_i})$

2. Cohomology ring of Bott manifolds

Let γ_i be the tautological line bundle

$$\gamma_i : \underline{\mathbb{C}} \oplus \xi_i \longrightarrow P(\underline{\mathbb{C}} \oplus \xi_i) = B_i$$

Let $x_i := c_1(\gamma_i^*) = -c_1(\gamma_i) \in H^2(B_i)$

$$\Rightarrow \quad H^*(B_n) \cong H^*(B_{n-1})[x_n] / \langle x_n^2 + c_1(\xi_n) x_n \rangle$$

$$\cong \mathbb{Z}[x_1, \dots, x_n] \Big/ \begin{matrix} x_1^2 \\ x_2^2 + c_1(\xi_2) x_2 \\ \vdots \\ x_n^2 + c_1(\xi_n) x_n \end{matrix}$$

$c_1(\xi_i) \in H^2(B_{i-1})$

$\therefore c_1(\xi_i) = c_{i1} x_1 + \cdots + c_{i\,i-1} x_{i-1}$

for some $c_{ij} \in \mathbb{Z}$

$$\cong \mathbb{Z}[x_1, \dots, x_n] \Big/ \begin{matrix} x_1^2 \\ x_2(c_{21} x_1 + x_2) \\ \vdots \\ x_n(c_{n1} x_1 + \cdots + c_{n,n-1} x_{n-1} + x_n) \end{matrix}$$

3. Cohomological rigidity question for Bott manifolds

(Question) M, N : n-stage Bott manifold

$H^*(M;\mathbb{Z}) \cong H^*(N;\mathbb{Z})$ as graded rings

$\overset{?}{\Longrightarrow}$ $M \cong N$ (homeomorphic, or diffeomorphic) ?

If so, we say Bott manifolds are cohomologically rigid.

[Masuda-Panov, 2008 Sbornik Math.]

M : n-stage Bott manifold

If $H^*(M;\mathbb{Z}) \cong H^*((\mathbb{C}P^1)^n)$ trivial n-stage Bott manifold

$\Longrightarrow$ $M \underset{\text{diffeom}}{\cong} (\mathbb{C}P^1)^n$

[Choi-Masuda-S, 2010 Trans. AMS]

M, N : any 3-stage Bott manifold with

$H^*(M;\mathbb{Z}) \cong H^*(N;\mathbb{Z}) \implies M \underset{\text{diffeom}}{\cong} N$.

Theorem 1

only one stage of the Bott tower sequence is a nontrivial fibration

Any 1-twist Bott manifolds are cohomologically rigid.

i.e. if M, N : 1-twist Bott mfds with $H^*(M) \cong H^*(N)$

$\implies M \cong N$ (diffeom)

4. Toric and Quasitoric manifolds

Toric variety = a normal complex algebraic variety of $\dim_{\mathbb{C}}$ n with $(\mathbb{C}^*)^n$-action having a dense orbit.

Toric manifold = a nonsingular toric variety

Quasitoric manifold = a closed 2n-dim manifold M with locally standard $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}$ action &

$\exists$ a map $\pi : M^{2n} \longrightarrow P^n$: simple convex polytope such that $\pi^{-1}(pt) = T^n$-orbit. In this case we say M is over P.

Standard action of T^n on $\mathbb{C}^n$:

$$(t_1,\dots,t_n)\cdot(z_1,\dots,z_n) = (t_1z_1,\dots,t_nz_n)$$

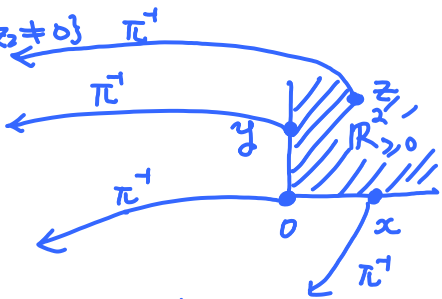
for $(t_1,\dots,t_n)\in T^n$, $(z_1,\dots,z_n)\in\mathbb{C}^n$

$$\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^n/T^n \cong \mathbb{R}^n_{\geq 0}$$

$\pi^{-1}(z)=\{(z_1,z_2)\mid z_1\neq 0, z_2\neq 0\}$ (T^2 acts freely)

$\pi^{-1}(y)=\{(0,z_2)\mid z_2\neq 0\}$ fixed by $S^1\times 1$

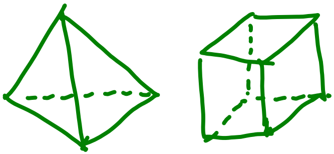
$\pi^{-1}(0)=\{(0,0)\}$ fixed by T^2



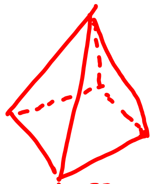
$\pi^{-1}(x)=\{(z_1,0)\mid z_1\neq 0\}$: fixed by $1\times S^1$

Simple convex polytope of dim n

= polytope s.t at each vertex exactly n facets are intersecting



simple



not simple

For a q.t mfd $M \xrightarrow{\pi} P$,

$\pi^{-1}(\text{vertex}) \in M^{T^n}$: fixed point set

$\pi^{-1}(\text{facet}) \subset$ fixed set by a circle subgroup of T^n.

Example

(1) $\mathbb{C}P^n$: toric manifold (and q.t. mfd over Δ^n)

(2) Bott manifold B_n : toric mfd (q.t mfd over I^n)

(3) Any proj. toric manifolds $\underset{\not\Leftarrow}{\Rightarrow}$ q.t. mfd

$\mathbb{C}P^1 \# \mathbb{C}P^1$ is a q.t mfd over I^n, but it does not have a compatible almost complex structure.

$\therefore$ not a toric manifold.

$\boxed{\text{(Question) Is the class of toric manifolds (or q.t. mfds) cohomologically rigid?}}$

5 BQ-algebra

BQ-algebra of rank n over R

= graded algebra over R gen. by $x_1, \cdots, x_n$

with $\deg x_i = 2$ such that

(1) $x_k^2 = \sum_{i<k} c_{ik} x_i x_k$, $c_{ik} \in R$ for $1 \leq k \leq n$

(2) $\prod_{i=1}^{n} x_i \neq 0$

Cohomological complexity of a BQ-algera

= the number of k's s.t $x_k^2 \neq 0$.

[actually this number depends on choices of generators $x_1, \cdots, x_n$.
So, the cohomological complexity should be the minimum of the
previousely defined number among all choices of generators.]

For a Bott manifold B_n, its cohomology $H^*(B_n;\mathbb{Z})$ is a BQ-algebra over $\mathbb{Z}$.

Is the converse true?

Theorem 2

M^{2n} : quasitoric manifold over a simple convex poly. P.

If $H^*(M)$ is a BQ-algebra over $\mathbb{Z}$,

$\Rightarrow$ (1) $P = I^n$ (by the work of Masuda - Panov)

(2) M is homeom to an n-stage Bott mfd.

6 Twist number and cohomological complexity

For an n-stage Bott manifold

$$B_n \longrightarrow B_{n-1} \longrightarrow \cdots\cdots B_i \longrightarrow B_{i-1} \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0$$

The **twist number** is the number of non-trivial fibrations in the tower.

For a Bott manifold B_n, there may be another tower

$$B_n \longrightarrow B_{n-1}' \longrightarrow \cdots \longrightarrow B_i' \longrightarrow B_{i-1}' \longrightarrow \cdots \longrightarrow B_1' \longrightarrow B_0 = pt$$

However we can show that the twist number is independent of a choice of a tower.

For a Bott manifold B_n, we can see easily that

twist number of B_n $\geq$ cohom. complexity of $H^*(B_n)$

But they are actually equal, i.e.,

Theorem 3

For a Bott manifold B_n

twist number of B_n $=$ cohom. complexity of $H^*(B_n)$.

7. Cohomological rigidity of q.t manifolds over I^n.

Theorem 4

M^{2n}, N^{2n} : quasitoric manifolds s.t $H^*(M)$ and $H^*(N)$ are BQ-algebras over $\mathbb{Z}$ with cohomological complexities equal to 1.

Then $H^*(M) \approx H^*(N) \Rightarrow M \approx N$ (homeom)

(proof) By Theorem 2 M and N are homeom to n-stage Bott manifolds B_n and B_n', resp. By Theorem 3 their twist numbers equal to 1. $\therefore$ By Theorem 1

$B_n \overset{\text{diff}}{\approx} B_n'$. $\Rightarrow$ $M \overset{\text{homeo}}{\approx} N$.

Theorem 5

M^6, N^6 : 6-dim quasitoric manifolds s.t

$H^*(M)$ and $H^*(N)$ are isomorphic BQ-algebras over $\mathbb{Z}$

$\Rightarrow M \approx N$ (homeom)

(proof) By Theorem 2 $M \overset{\text{homeo}}{\approx} B_3$, $N \overset{\text{homeo}}{\approx} B_3'$

for some 3-stage Bott manifolds B_3 and B_3'.

By [Choi-Masuda-S 2010] $B_3 \underset{\text{diffeom}}{\approx} B_3'$

$\therefore M \underset{\text{homeo}}{\approx} N$

8 Cohomology with $\mathbb{Z}_{(2)}$ coefficients

Theorem 2 ~ 5 are still valid with $\mathbb{Z}_{(2)}$-coefficients.

Theorem 2' M^{2n}: q.t mfd over P

If $H^*(M:\mathbb{Z}_{(2)})$ is a BQ-algebra over $\mathbb{Z}_{(2)}$, then

(1) $P = I^n$ (2) $M \approx B_n$ (homeom)

Theorem 3' For a Bott manifold B_n

twist number of B_n = cohomological complexity of $H^*(M:\mathbb{Z}_{(2)})$

Theorem 4' M, N: q.t mfd s.t $H^*(M;\mathbb{Z}_{(2)})$ and $H^*(N:\mathbb{Z}_{(2)})$ are BQ-algebras over $\mathbb{Z}_{(2)}$ with cohomological complexity $= 1$.

$\Rightarrow$ $M \approx N$ homeomorphic

Theorem 5' M, N: 6-dim q.t mfd s.t $H^*(M;\mathbb{Z}_{(2)})$ and $H^*(N;\mathbb{Z}_{(2)})$ are BQ-algebras over $\mathbb{Z}_{(2)}$ $\Rightarrow$ $M \approx N$ homeom

9. Sketch of proof of Theorem 1

Lemma 6 If a Bott manifold B_n has a one-twist Bott tower structure $B_n \to B_{n-1} \to \cdots \to B_0$, then it has another Bott tower structure $B_n \to B_{n-1}' \to \cdots \to B_0'$ whose last stage is nontrivial and all other stages are trivial. $\square$

Let B_n be a one-twist Bott manifold, then we may assume that $B_{n-1} = (\mathbb{C}P^1)^n$ i.e.,

$$\underbrace{B_n \longrightarrow B_{n-1}}_{\text{nontrivial fibration}} \underbrace{\longrightarrow \cdots \longrightarrow B_i \to B_{i-1} \longrightarrow \cdots \longrightarrow B_1 \to B_0}_{\text{trivial fibration}}$$

$\therefore\ H^*(B_{n-1}) \cong \mathbb{Z}[x_1, \ldots, x_{n-1}] / \langle x_i^2 = 0,\ i = 1, \ldots, n-1 \rangle$

Let $B_n = P(\mathbb{C} \oplus \gamma^\alpha)$ where γ^α is the line bundle over B_{n-1} with

$$c_1(\gamma^\alpha) = \alpha = \sum_{i=1}^{n-1} a_i x_i \in H^*(B_{n-1})$$

For notational convenience, let $M_\alpha := P(\mathbb{C} \oplus \gamma^\alpha)$

$$\Rightarrow \quad H^*(M_\alpha) \cong \mathbb{Z}[x_1, \dots, x_{n-1}, y_\alpha] / \langle x_1^2, \dots, x_{n-1}^2, y_\alpha^2 + \alpha y_\alpha \rangle$$

Lemma 8 The following are equivalent

(1) $H^*(M(\alpha) : \mathbb{Q}) \cong H^*((\mathbb{CP}^1)^n : \mathbb{Q})$

(2) $\exists\ u \in H^*(B_{n-1} : \mathbb{Q})$ s.t. $(y_\alpha + u)^2 = 0$

(3) $\alpha = a_i x_i$ for some $i = 1, \dots, n-1$.

Moreover there are two diffeomorphism types in this case

$M(\alpha) \cong (\mathbb{CP}^1)^n$ or $M(\alpha) = (\mathbb{CP}^1)^{n-1} \times \mathcal{H}$ ($\mathcal{H}$ is a Hirzebruch surface),

and $H^*(M(\alpha)) \cong H^*((\mathbb{CP}^1)^n) \iff a_i = \text{even}$ in (3)

Now consider $M(\alpha)$ and $M(\beta)$ s.t

$$H^*(M(\alpha)) \xrightarrow[\cong]{\phi} H^*(M(\beta)).$$

By some argument with Lemma 8 (3), we can show that

(i) ϕ preserves the subring $H^*((\mathbb{C}P^1)^{n-1})$

(ii) $\phi(\alpha) \equiv \beta \pmod 2$

(iii) $\phi(\alpha^2) = \beta^2$

[we may need to alter the original isomorphism by composing ϕ with an automorphism of $H^*(M(\beta))$ fixing the subring $H^*((\mathbb{C}P^1)^{n-1})$.]

Further it is easy to prove that any autom on $H^*(B_{n-1}) = \mathbb{Z}[x_1, \cdots, x_{n-1}]/\langle x_i^2 \mid i = 1, \ldots, n-1 \rangle$ is generated by permutations of the generators up to sign.

$\therefore$ We may assume that the isomorphism

$\phi : H^*(M(\alpha)) \longrightarrow H^*(M(\beta))$ is

(1) the identity on the subring $H^*(B_{n-1})$;

(2) $\alpha \equiv \beta \pmod 2$, and

(3) $\alpha^2 = \beta^2$

$\alpha \equiv \beta\ (2) \implies \alpha - \beta = 2\omega$ for some $\omega \in H^*(B_{n-1})$

Let

$$\xi_1 = \gamma^\alpha \oplus \mathbb{C}$$
$$\xi_2 = \gamma^{-\omega}(\gamma^\beta \oplus \mathbb{C})$$

$\Rightarrow$ $c_1(\xi_1) = \alpha = \beta + 2\omega = c_1(\xi_2)$

$c_2(\xi_1) = 0$

$c_2(\xi_2) = -\omega(\beta - \omega) = 0$ because $\alpha^2 = \beta^2$

We now need the following lemma:

Lemma 9

Let $\xi_1 = \gamma^{\alpha_1} \oplus \gamma^{\alpha_2}$, $\xi_2 = \gamma^{\beta_1} \oplus \gamma^{\beta_2}$ be sums of two $\mathbb{C}$-line bundles over $(\mathbb{C}P^1)^k$ s.t.

$$c_1(\xi_1) = c_1(\xi_2) \quad \text{and} \quad c_2(\xi_1) = c_2(\xi_2) = 0 .$$

$\Rightarrow \quad \xi_1 \cong \xi_2$ $\square$

By this lemma $\xi_1 \cong \xi_2$.

$$\begin{aligned} \Rightarrow \quad M(\alpha) &= p(\xi_1) \\ &\cong p(\xi_2) \\ &\cong p(\gamma^{-\omega}(\gamma^{\beta} \oplus \mathbb{C})) \\ &\cong p(\gamma^{\beta} \oplus \mathbb{C}) = M(\beta) \end{aligned}$$

10. Some comments on proof of Theorem 2

M: quasitoric manifold over P

Suppose $H^*(M)$ is a BQ-algebra over $\mathbb{Z}$.

$\Rightarrow$ $H^*(M:\mathbb{Z}_2)$ is a BQ-algebra over $\mathbb{Z}_2$

By a result of Masuda-Panov (2008)

P is combinatorially equivalent to I^n.

(such property is called the cohomological rigidity of polytopes)

Now by the result of Dobrinskaya (also CMS),

it is enough to show that all principal minors of the associated $n \times n$ matrix of M are 1.

11. Sketch of proof of Theorem 3

Let $B_n \longrightarrow B_{n-1} \to \cdots \to B_1 \to B_0$ be a Bott tower with the twist number t.

By a similar result similar to Lemma 6, we may assume that

$$B_{n-t} \longrightarrow B_{n-t-1} \longrightarrow \cdots \to B_1 \to B_0$$

is a trivial Bott tower.

$$\Rightarrow \quad H^*(B_n) \cong \mathbb{Z}[x_1, \cdots, x_n] \Big/ \left\langle \begin{array}{l} x_1^2 = \cdots = x_{n-t}^2 = 0 \\ x_i^2 + f_i x_i = 0 \text{ for } n-t+1 \leq i \leq n \end{array} \right\rangle$$

where $f_i = \sum_{j=1}^{i-1} c_{ij} x_j$

Now suppose the cohomological complexity of $H^*(B_n) = s < t$.

$$\Rightarrow \exists\ \psi : \mathbb{Z}[x_1, \cdots, x_n] \Big/ \begin{matrix} x_1^2 = \cdots = x_{n-t}^2 = 0 \\ x_i^2 + f_i x_i = 0,\quad n-t+1 \leq i \leq n \end{matrix}$$

$$\longrightarrow \mathbb{Z}[y_1, \cdots, y_n] \Big/ \begin{matrix} y_1^2 = \cdots = y_{n-s}^2 = 0 \\ y_j^2 + g_j y_j = 0,\quad n-s+1 \leq j \leq n \end{matrix}$$

Lemma 10 $\exists\ m$ $(n-t+1 \leq m \leq n)$ s.t

$f_m \equiv 0 \pmod 2$ and $f_m^2 = 0 \in H^*(B_{m-1})$ $\square$

Thus $f + 2\omega = 0$ for some $\omega \in H^2(B_{m-1})$

$$\Rightarrow \quad c(\gamma^{\omega} \oplus \gamma^{f_m + \omega}) = (1+\omega)(1+f_m+\omega)$$

$$= 1 + (f_m + 2\omega) - \frac{f_m^2}{4}$$

$$= 1$$

Thus $\gamma^{\omega} \oplus \gamma^{f_m + \omega}$ is trivial by the following lemma.

Lemma 11. A sum of two line bundles over a Bott manifold is trivial $\Leftrightarrow$ the total Chern class is trivial $\square$

Now $P(\mathbb{C} \oplus \gamma^{fm}) = P(\gamma^{\omega} \otimes (\mathbb{C} \oplus \gamma^{fm}))$

$$= P(\gamma^{\omega} \oplus \gamma^{fm+\omega})$$

$$= B_{m-1} \times \mathbb{C}P^1$$

$\therefore$ We can reduce the twist number of B_n to $t-1$, which is a contradiction.

$\therefore \quad s = t$ $\square$

Sketch of a different proof (More conceptual): by Masuda

$E \to X$: a complex 2-dim vector bundle,
$P(E) \xrightarrow{\pi} X$: the corresponding $\mathbb{C}P^1$-bundle

$T_f P(E) \longrightarrow P(E)$: tangent bundle of $P(E)$ along fibers of $P(E) \longrightarrow X$.

$E' \longrightarrow X$: another rank 2 v.b.

If
$$\begin{tikzcd} P(E) \arrow[rr, "\phi", "\cong"'] \arrow[dr] & & P(E') \arrow[dl] \\ & X & \end{tikzcd}$$
fiber bundle isom

$\Rightarrow$
$$\begin{tikzcd} T_f P(E) \arrow[r, "\cong"] \arrow[d] & T_f P(E') \arrow[d] \\ P(E) \arrow[r, "\phi", "\cong"'] & P(E') \end{tikzcd}$$
: v.b. isomorphism

Hence $\quad p_1(T_f P(E)) = p_1(T_f P(E'))$

& $\quad w_2(T_f P(E)) = w_2(T_f P(E'))$

Lemma 12 For $\pi: P(E) \longrightarrow X$ as above

$$p_1(T_f P(E)) = c_1(E)^2 - 4c_2(E)$$

$$w_2(T_f P(E)) = \pi^*(w_2(E))$$

- In particular when X is a Bott manifold & $P(E) = P(\mathbb{C} \oplus \gamma^{\alpha_i}) \longrightarrow X$ is trivial

$\Rightarrow \quad c_1(E)^2 - 4c_2(E) = \alpha_i^2 = 0$

& $\quad w_2(E) \equiv \alpha_i \equiv 0 \pmod 2$

■ Conversely, if $\alpha_i^2 = 0$ and $2 \mid \alpha_i$, then $P(\mathbb{C} \oplus \gamma^{\alpha_i}) \longrightarrow X$ is a trivial fibration.

(proof) $\quad P(\mathbb{C} \oplus \gamma^{\alpha_i}) \cong P(\gamma^{-\frac{1}{2}\alpha_i}(\mathbb{C} \oplus \gamma^{\alpha_i}))$

$$= P(\gamma^{-\frac{\alpha_i}{2}} \oplus \gamma^{\frac{\alpha_i}{2}})$$

But $c(\gamma^{-\frac{\alpha_i}{2}} \oplus \gamma^{\frac{\alpha_i}{2}}) = (1 - \frac{\alpha_i}{2})(1 + \frac{\alpha_i}{2}) = 1$.

Thefore $\gamma^{-\frac{\alpha_i}{2}} \oplus \gamma^{\frac{\alpha_i}{2}}$ is a trivial bundle by the following:

Lemma 13 If $E \to X$ is a sum of two line bundles over a Bott manifold with $c(E) = 1$, then E is trivial.

$\therefore$ We need to consider the elements $\alpha \in H^2(B_m)$ at $\alpha^2 = 0$ & $2 \mid \alpha$.

In fact we can show that if

$$H^*(B_n;\mathbb{Z}) \cong \mathbb{Z}[x_1,\cdots,x_n] \big/ \langle x_i^2 + \alpha_i x_i \mid i=1,\cdots,n\rangle$$

then

the twist number of B_n

$= \quad n - |\{i \mid \alpha_i^2 = 0 \;,\; 2 \mid \alpha_i\}|$

$=$ the cohomological complexity of B_n

$\square$