On isovariant Hopf type theorems

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20091212

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Hopf's classification theorem

At the beginning,

let us recall famous

Hopf's classification theorem

Hopf's classification theorem

M	:	a connected, orientable, closed n-manifold
S^n	:	an n-dimensional sphere $(n \geqq 1)$
$[M, S^n]$	:	the set of homotopy classes of continuous maps $f : M \to S^n$

Then, the degree function

$$\deg : [M, S^n] \to \mathbb{Z}$$

is bijective.

Equivariant versions

G : a finite group
S^n : an n-sphere with free G-action $(n \geqq 1)$

Then,

1. The degree function $\deg : [S^n, S^n]_G \to \mathbb{Z}$ is injective.

2. $\mathrm{Im}(\deg) = 1 + |G|\mathbb{Z}$.

(See tom Dieck's book)

An equivariant Hopf theorem

As a corollary, we have:

Put

$$D([f]) = (\deg f - 1)/|G|.$$

Then,under the same assumptions as above, the map

$$D : [S_n, S_n]_G \to \mathbb{Z}$$

is a bijection.

Isovariant maps

Next, we recall isovariant maps.

Def. of an isovariant map

$$\begin{aligned} G &: \text{ a group} \\ X, Y &: G\text{-spaces} \\ \varphi : X \to Y &: \text{ a } G\text{-equivariant map} \end{aligned}$$

φ is a G-*isovariant map*

$\overset{\text{def}}{\Longleftrightarrow}$ if it preserves the isotropy groups, that is, $G_x = G_{\varphi(x)}$ holds for all $x \in X$.

Isovariant homotopy

· A G-homotopy $F : X \times I \to Y$ is called a G-*isovariant homotopy*

$$\overset{\text{def}}{\Longleftrightarrow} F \text{ is } G\text{-isovariant.}$$

· $[X, Y]_G^{\text{isov}}$: the set of isovariant homotopy classes of isovariant maps $f : X \to Y$

What is Today's talk?

Now,we explain about today's talk

Our settings

We will consider the structure of $[M, SW]_G^{\mathrm{isov}}$ under the following setting:

G	:	a finite group
M	:	a connected, orientable, closed free G-manifold
SW	:	a faithful unitary G-representation sphere

and,

The Borsuk-Ulam inequality

in the following page.

The Borsuk-Ulam inequality

$$\dim M + 1 \leqq \dim SW - \dim SW^{>1} \qquad \text{(BUI)}$$

where $SW^{>1}$ is the **singular set**, i.e.,

$$SW^{>1} \stackrel{\text{def}}{=} \bigcup_{\{1\} \neq H \leq G} SW^H$$

$$= \{x \in SW \mid G_x \neq 1\}$$

where if $SW^{>1} = \emptyset$, we put $\dim SW^{>1} = -1$.

Why is it called "Borsuk-Ulam" ineq.?

Beacause it appears in the following isovariant Borsuk-Ulam theorem:

M : an m-dimensional $mod|G|$-homology sphere on which G acts freely.

SW : a G-representation sphere

$\exists$ a G-isovariant map $f : M \to SW \Longrightarrow$

$$\dim M + 1 \leqq \dim SW - \dim SW^{>1}.$$

||

$$[M, SW]_G^{\text{isov}} = [M, SW_{\text{free}}]_G$$

Put

$$SW_{\text{free}} = SW \setminus SW^{>1}.$$

Then, by assumption, we have :

$$[M, SW]_G^{\text{isov}} = [M, SW_{\text{free}}]_G.$$

In our setting, the problem was reduced to the study of the set of equivariant homotopy classes of equivariant maps $f : M \to SW_{\text{free}}$, that is,.

Our Problem (Again)

Determine the set $[M, SW]_G^{\mathrm{isov}}$ under the inequality

$$\dim M + 1 \leqq \dim SW - \dim SW^{>1} \qquad \text{(BUI)}$$

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Determine the set $[M, SW]_G^{\text{isov}}$ under the inequality

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This problem is reduced to

Determine the set $[M, SW_{\text{free}}]_G$ under the inequality

$$\dim M + 1 \leqq \dim SW - \dim SW^{>1} \qquad \text{(BUI)}$$

Topology of SW_{free}, preparation

Recall that

$$SW^{>1} \stackrel{\text{def}}{=} \bigcup_{\{1\} \neq H \leq G} SW^H, \quad SW_{\text{free}} = SW \setminus SW^{>1}$$

Put

$$d = \dim SW - \dim SW^{>1}.$$

Since the representation is unitary, we have

$d \geqq 2,$ d is even.

Topology of $SW_{\rm free}$

1. $SW_{\rm free}$ is $(d-2)$-connected.

2. Put

$$\mathcal{A} = \{H \in \mathrm{Iso}(W) \,|\, \dim SW^H = \dim SW^{>1}\},$$

then

$$\pi_{d-1}(SW_{\rm free}) \cong \bigoplus_{H \in \mathcal{A}} \mathbb{Z}$$

Outline of Proof of (1)

By general position arguments.

Outline of Proof of (2)

We note that $\dim S(W^H)^\perp = d - 1$ for $H \in \mathcal{A}$.

By the Hurewicz theorem and the Mayer-Vietoris exact sequence, we have the following composite of isomorphisms, where i,j and i_H are inclusions.

$(\mathcal{A} = \{H \in \mathrm{Iso}(W) | \dim SW^H = \dim SW^{>1}\})$

$$\begin{aligned}\pi_{d-1}(SW_{\text{free}}) &\cong H_{d-1}(SW_{\text{free}};\mathbb{Z})\\ &\xrightarrow{i_*} H_{d-1}(SW_{\mathcal{A}-\text{free}};\mathbb{Z})\\ &\xrightarrow{j_*} \bigoplus_{H\in\mathcal{A}} H_{d-1}(SW\setminus SW^H;\mathbb{Z})\\ &\xleftarrow{\oplus i_{H*}} \bigoplus_{H\in\mathcal{A}} H_{d-1}(S(W^H)^\perp;\mathbb{Z})\\ &\cong \bigoplus_{H\in\mathcal{A}} \mathbb{Z}\end{aligned}$$

Another description

Since G acts on SW_{free}, $\pi_{d-1}(SW_{\text{free}})$ and $H_{d-1}(SW_{\text{free}};\mathbb{Z})$ are regarded as $\mathbb{Z}G$-modules.

For $H \in \mathcal{A}$,

$$gS(W^H)^\perp = S(W^{gHg^{-1}})^\perp \quad (g \in G)$$

and then,we have

$$gS(W^H)^\perp = S(W^H)^\perp \Leftrightarrow g \in NH.$$

$\mathbb{Z}G$-isomorphisms

Set $\mathcal{A}/G = \{(H) \mid H \in \mathcal{A}\}$.

Then, we have the following $\mathbb{Z}G$-isomorphisms

$$\Psi : H_{d-1}(SW_{\mathrm{free}}; \mathbb{Z}) \to \bigoplus_{(H)\in A/G} \mathbb{Z}[G/NH]$$

$$\Psi \circ h : \pi_{d-1}(SW_{\mathrm{free}}) \to \bigoplus_{(H)\in A/G} \mathbb{Z}[G/NH]$$

Equivariant cohomology

The equivariant cochain complex is defined by

$$C^*_G(M;\pi) \overset{\text{def}}{=} \mathrm{Hom}_{\mathbb{Z}G}(C_*(M);\pi), \delta = \mathrm{Hom}_{\mathbb{Z}G}(\partial).$$

where π is a $\mathbb{Z}G$-module. The equivariant cohomology $\mathfrak{H}^*_G(M;\pi)$ is defined by

$$\mathfrak{H}^*_G(M;\pi) = H^*(C^*_G(M;\pi),\delta)$$

It is the main tool for our approach.

Theorem (Existence of isovariant maps)

If the inequality

$$\dim M + 1 \leqq \dim SW - \dim SW^{>1} \qquad \text{(BUI)}$$

holds, there exists a G-isovariant map $f : M \to SW$.

Proof

Since $\dim M \leqq d-1$ and SW_{free} is $(d-2)$-connected,

$$\mathfrak{H}_G^*(M;\pi_{*-1}(SW_{\mathrm{free}}))=0.$$

Equivariant obstruction theory says that it means that there exists a G-map $f: M \to SW_{\mathrm{free}}$.

Classifying problem

At first, we will discuss this problem, from a cohomological approach.

Equivariant cohomology as an obstruction

Set $m = \dim M$ and $\pi_m = \pi_m(SW_{\text{free}})$.

Let $f, g : M \to SW_{\text{free}}$ be G-maps.

Generally,speaking,
the equivariant obstruction class $\gamma_G(f, g)$ to the existence of a G-homotopy between $f$and g lies in

$$\mathfrak{H}^*_G(M; \pi_*).$$

But,in this case...

Since

SW_{free} is $(d-2)$-connected,

$m \leqq d-1$,

$$\gamma_G(f,g) \in \mathfrak{H}_G^m(M;\pi_m).$$

Theorem (strict case)

If $\dim M < d-1$,
then all isovariant maps from M to SW are isovariantly homotopic each other.

Proof.

It follows from

$$\mathfrak{H}_G^m(M;\pi_m) = 0.$$

The Main Stage

From now, we assume that

$$\dim M + 1 = d \overset{\text{def}}{=} \dim SW - \dim SW^{>1} \qquad \text{(BUE)}$$

(In this case, by assumption $\dim M$ is odd.)

Correspondence

Fix an isovariant map $f_0 : M \to SW$, which is called a reference map. Then according to the equivariant obstruction theory, the correspondence

$$\gamma_{f_0} : [M, SW_{\mathrm{free}}]_G \longrightarrow \mathfrak{H}_G^{d-1}(M; \pi_{d-1})$$

defined by

$$[f] \mapsto \gamma_G(f_0, f)$$

is a bijection.

orientation homomorphism $w : G \to \{\pm 1\}$

Put

$$w(g) = 1 \text{ or } -1$$

if the action of g preserves the orientation of M or not respectively.

Define a $\mathbb{Z}G$-module $\mathbb{Z}_w$ by

$$\begin{array}{rcl} \text{underlying module} & : & \mathbb{Z} \\ G\text{-action} & : & g \cdot k = w(g)k \end{array}$$

Decompositon of $\mathcal{A}$

Put $K_w = \ker w$.

We decompose $\mathcal{A}$ into two parts as follows :

$$\mathcal{A}^+ \stackrel{\text{def}}{=} \{H \in \mathcal{A} \mid NH \leq K_w\}$$
$$\mathcal{A}^- \stackrel{\text{def}}{=} \{H \in \mathcal{A} \mid NH \not\leq K_w\}.$$

Set

$$\mathcal{A}^\pm/G \stackrel{\text{def}}{=} \{(H) \mid H \in \mathcal{A}^\pm\}$$

What is $\mathfrak{H}_G^{d-1}(M;\pi_{d-1})$?

Under the assumption,

$$\mathfrak{H}_G^{d-1}(M;\pi_{d-1}) \cong \bigoplus_{(H)\in A^+/G} \mathbb{Z} \oplus \bigoplus_{(H)\in A^-/G} \mathbb{Z}_2$$

Consequently under our setting, we have

$$[M, SW]_G^{\mathrm{isov}} \cong \bigoplus_{(H)\in A^+/G} \mathbb{Z} \oplus \bigoplus_{(H)\in A^-/G} \mathbb{Z}_2$$

Outline of pf.

$$
\begin{aligned}
\mathfrak{H}_G^{d-1}(M;\pi_{d-1}) &\cong \bigoplus_{(H)\in A/G} \mathfrak{H}_G^{d-1}(M;\mathbb{Z}[G/NH]) \\
&\cong \bigoplus_{(H)\in A/G} H^{d-1}(M;\{\mathbb{Z}[G/NH]\}) \\
&\cong \bigoplus_{(H)\in A/G} H_0(M;\{\mathbb{Z}_w[G/NH]\}) \\
&\cong \bigoplus_{(H)\in A/G} \mathbb{Z}_w[G/NH]/ < a - w(g)a >
\end{aligned}
$$

$$
\bigoplus_{(H)\in A^+/G} \mathbb{Z} \oplus \bigoplus_{(H)\in A^-/G} \mathbb{Z}_2
$$

When G is abelan . . .

Suppose that G is abelian.

If the action is ori.-pre.,then $\mathcal{A} = \mathcal{A}^+$. Hence,

$$[M, SW]_G^{\mathrm{isov}} \cong \bigoplus_{H \in \mathcal{A}} \mathbb{Z}$$

If the action is not ori.-pre.,then $\mathcal{A} = \mathcal{A}^+$. Hence,

$$[M, SW]_G^{\mathrm{isov}} \cong \bigoplus_{H \in \mathcal{A}} \mathbb{Z}_2$$

Towards a Hopf -type theorem

Hearafter, we will give another approach to the correspondence above discussed.

Decomposition of $H_{d-1}(SW_{\text{free}})$

Set

$$SW_{\mathcal{A}^+-\text{free}} = SW \setminus \bigcup_{H \in \mathcal{A}^+} SW^H$$

$$SW_{\mathcal{A}^--\text{free}} = SW \setminus \bigcup_{H \in \mathcal{A}^-} SW^H$$

Then, we have a decomposition ($\mathbb{Z}G$-iso):

$$H_{d-1}(SW_{\text{free}}) \cong_G H_{d-1}(SW_{\mathcal{A}^+-\text{free}}) \oplus H_{d-1}(SW_{\mathcal{A}^--\text{free}})$$

By identifing $H_{d-1}(SW_{\mathcal{A}^{\pm}-\mathrm{free}})$ with $\bigoplus_{(H)\in\mathcal{A}^{\pm}/G} \mathbb{Z}[G/NH]$

1. $\exists d_H(f) \in \mathbb{Z}$ such that

$$f_*^+([M]) = (d_H(f)\sigma_H)_{(H)\in\mathcal{A}^+/G},$$

where $\sigma_H = \sum_{\bar{a}\in G/NH} w(a)\bar{a}$

2. $f_*^-([M]) = 0.$

The Multidegree

The multidegree $\mathrm{mDeg}\, f$ of an isovariant map $f : M \to SW$ is defined by

$$\mathrm{mDeg}\, f = (d_H(f))_{(H)} \in \bigoplus_{(H)\in\mathcal{A}^+/G} \mathbb{Z}.$$

It is an isovariant homotopy invariant.

Main results

1. Let $f, g : M \to SW$ be G-isovariant maps, then

$$\mathrm{mDeg}\ f - \mathrm{mDeg}\ g \in \bigoplus_{(H)\in\mathcal{A}^+/G} |NH|\mathbb{Z}.$$

2. f_0 : a reference map.

$$\forall \alpha \in \bigoplus_{(H) \in \mathcal{A}^+/G} |NH|\mathbb{Z}, \exists f : S \to SW \text{ (a } |G|\text{-isov)}$$

such that

$$\text{mDeg } f - \text{mDeg } f_0 = \alpha$$

3. The number of G-isovariant homotopy classes with same multidegree is $2^{|\mathcal{A}^-/G|}$

If $\mathcal{A}^- = \emptyset$,

If $\mathcal{A}^- = \emptyset$, by our main theorem

$$\mathrm{mDeg} \ : [M, SW]_G^{\mathrm{isov}} \to \bigoplus_{(H)\in\mathcal{A}/G} \mathbb{Z}$$

is injective. For a fixed reference map f_0, we define $D_{f_0}(f)$ by

$$D_{f_0}(f) = (\frac{1}{|NH|}(d_H(f) - d_H(f_0)))_{(H)}$$

an Isovariant Hopf theorem

If $\mathcal{A}^- = \emptyset$,then

$$D_{f_0} : [M, SW]_G^{\text{isov}} \to \bigoplus_{(H) \in \mathcal{A}/G} \mathbb{Z}$$

is a bijection.
In particular if G acts orientation-preservingly on M, D_{f_0} is a bijection.

Thank you for your listenning.

That's all for today.

Be careful to influ.

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