Torus fibrations and localization of index

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Joint work in progress with Hajime Fujita and Mikio Furuta

1. H. Fujita, M. Furuta, Y, *Torus fibrations and localization of index I*, arXiv:0804.3258.
2. H. Fujita, M. Furuta, Y, *Torus fibrations and localization of index II*, arXiv:0910.0358.
3. M. Furuta, *Index theorem 1*, Translations of Mathematical Monographs, vol. 235, Amer. Math. Soc., Providence, RI, 2007.

Z_2-graded $Cl(TM)$-module bundle

M Riemannian manifold

Definition

A *$\mathbb{Z}_2$-graded $Cl(TM)$-module bundle* (W, c) *on* M is the following data

- $W = W^0 \oplus W^1$ as Hermitian vector bundles
- $c\colon TM \to \mathrm{End}_{\mathbb{C}} W$: $\mathbb{R}$-linear hom. s. t.
  - $W^i \ni w \mapsto c(u)w \in W^{i+1}$ (degree-one)
  - $(c(u)w_1, w_2)_W = -(w_1, c(u)w_2)_W$ (skew-Hermitian)
  - $c(u) \circ c(v) + c(v) \circ c(u) = -2(u, v)_M \, \mathrm{id}_W$.

Example (Exterior algebra bundle)

- $W = \wedge^{\bullet} T^*M = \wedge^{\mathrm{even}} T^*M \oplus \wedge^{\mathrm{odd}} T^*M$
- $c(u) = u^{\wedge} - u^{\llcorner}$

Dirac-type operator

(W, c) Z_2-graded $Cl(TM)$-module bundle on M

Definition

A 1st-order linear differential operator $D\colon \Gamma(W) \to \Gamma(W)$ is of *Dirac-type* if

- $\Gamma(W^i) \ni s \mapsto Ds \in \Gamma(W^{i+1})$ (degree-one)
- $D^* = D$ (formally self-adjoint), where D^* is defined by

$$\int_M (Ds_1, s_2)_W dvol = \int_M (s_1, D^* s_2)_W dvol \quad \forall s_i \in \Gamma_c(W)$$

- $\sigma(D) = c$, where $\sigma(D)\colon T^*M(\cong TM) \to \mathrm{End}_{\mathbb{C}} W$ is the principal symbol.

Fact & Definition (index)

If M is closed, then a Dirac-type operator D is Fredholm, i.e. $\dim\ker D < +\infty$.

$$\operatorname{ind} D := \dim\ker D^0 - \dim\ker D^1 \in \mathbb{Z} \quad (D^k = D|_{W^k})$$

- $\operatorname{ind} D$ is invariant under continuous deformations.

Dirac-type operator

Examples

1. *de Rham operator* $D = d + d^* \colon \Gamma(\wedge^\bullet T^*M) \to \Gamma(\wedge^\bullet T^*M)$

$$\operatorname{ind} D = \sum_i (-1)^i \dim \mathrm{H}^i(M; \mathbb{R}).$$

2. *Dolbeault operator* $D = \sqrt{2}(\bar\partial + \bar\partial^*) \colon \Gamma(\wedge^\bullet T^*M^{0,1}) \to \Gamma(\wedge^\bullet T^*M^{0,1})$

$$\operatorname{ind} D = \sum_i (-1)^i \dim \mathrm{H}^i(M; \mathcal{O}_M).$$

- $\ker D \cong \mathrm{H}^\bullet(M)$ ($\because$ Hodge theory)

Localization

When M is closed, ind D is sometimes determined by the information on a specific subset under an appropriate geometric condition.

Examples

1. **Poincaré-Hopf's theorem.** For $X \in \Gamma(TM)$ with $X \pitchfork 0$,

$$\mathrm{ind}(d + d^*) = \sum_{p \in X^{-1}(0)} \mathrm{sign}\left(\det(dX_p)\right).$$

2. **Lefschetz formula for *G*-actions.** For $g \in G$ with $\# M^g < \infty$,

$$\mathrm{ind}_g D = \sum_{p \in M^g} \frac{\mathrm{tr}(g|W_p^0) - \mathrm{tr}(g|W_p^1)}{\det(1 - g|T_pM)}.$$

3. **Witten deformation.** For $0 \leq t$ & “some $h \in \mathrm{End}(W)$",

$$\mathrm{ind}\, D = \mathrm{ind}(D + th) \to \mathit{supp}\ h \quad (t \to +\infty).$$

Purpose

We give

1. a framework of a localization for the index of a Dirac-type operator on an open manifold
2. applications to Hamiltonian torus actions and Lagrangian fibrations

Main Theorem

W $\mathbb{Z}_2$-graded $Cl(TM)$-module bundle

$\downarrow$

M Riemannian manifold (possibly non-compact)

$\cup$

V open set s.t.

- $M \setminus V$ compact
- $\exists$"*acyclic compatible fibration* $\{(\pi_\alpha, D_\alpha)\}_{\alpha \in A}$"

Theorem (Fujita-Furuta-Y '09)

$\exists\,\mathrm{ind}(M, V) = \mathrm{ind}(M, W, V, \{(\pi_\alpha, D_\alpha)\}_{\alpha \in A}) \in \mathbb{Z}$ *(local index) satisfying*

1. *For M closed,*
$$\mathrm{ind}(M, V) = \mathrm{ind}\, D$$
2. *For $M' \subset M$ admissible open neighborhood of $M \setminus V$,*
$$\mathrm{ind}(M, V) = \mathrm{ind}(M', M' \cap V) \quad \textit{(excision)}$$
3. *For $M = M_1 \sqcup M_2$*
$$\mathrm{ind}(M, V) = \mathrm{ind}(M_1, M_1 \cap V) + \mathrm{ind}(M_2, M_2 \cap V) \quad \textit{(sum formula)}$$

- 2, & 3 $\Rightarrow$ For $V = M$, $\mathrm{ind}(M, V) = 0$ (vanishing)

Main Theorem

Under the above assumption, suppose

- M closed
- $M = \exists \sqcup_{i=1}^{k} O_i \cup V$ open covering.

Corollary (Localization)

$$\operatorname{ind} D = \sum_{i=1}^{k} \operatorname{ind}(O_i, O_i \cap V)$$

Motivation

- For a non-singular projective toric variety M^n with ample line bundle L and moment map $\pi\colon M \to \mathfrak{t}^*(=\mathrm{Lie}(T)^*)$,

$$\mathrm{ind}\left(\sqrt{2}(\bar{\partial}\otimes L+\bar{\partial}^*\otimes L)\right)=\#\pi(M)\cap\mathfrak{t}^*_{\mathbb{Z}} \text{ (Danilov '78).}$$

Such kinds of phenomena have been observed for

- Gelfand-Cetlin completely integrable system on the complex flag manifold (Guillemin-Sternberg '83)
- Goldman's completely integrable system on the moduli space of flat $SU(2)$-bundles on a Riemann surface (Goldman '86, Jeffrey-Weitsman '92)
- Pre-symplectic toric manifolds (Karshon-Tolman '93)
- Torus manifolds (Masuda '99, Hattori-Masuda '03)
- $\cdots$

Motivation

Understand these phenomena from the viewpoint of Witten deformation.

Witten deformation

$$\begin{tikzcd}(W, c) \arrow[d] & \mathbb{Z}_2\text{-graded } Cl(TM)\text{-module bundle} \\ M & \text{complete Riemannian manifold}\end{tikzcd}$$

For $t \geq 0$ define

$$D_t := D + th,$$

where $h \in \mathrm{End}(W)$ satisfying

- Hermitian
- degree-one
- $\mathrm{supp} h := \{x \in M \mid \ker(h_x\colon W_x \to W_x) \neq 0\}$ is compact
- $h \circ c + c \circ h = 0$

Point

1. ind D_t *is defined independently of* $\forall t \gg 0$ *in an appropriate sense.*
2. *“*ind D_t *is localized at* supp$h$ *as* $t \to +\infty$*”.*

Witten deformation

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- In our case what should we take as h?

Witten deformation

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Point

1. ind D_t *is defined independently of* $\forall t \gg 0$ *in an appropriate sense.*
2. *"*ind D_t *is localized at* supp$h$ *as* $t \to +\infty$*".*

- In our case what should we take as h? → "acyclic compatible fibration"

Observation - $\mathbb{C}P^1$ case

$$L = (H, \nabla)^{\otimes 2} \qquad W = \wedge^{\bullet} T^* M^{0,1} \otimes L$$

$$\downarrow \qquad D = \sqrt{2}(\bar\partial \otimes L + \bar\partial^* \otimes L) \colon \Gamma(W) \circlearrowleft$$

$$M = (\mathbb{C}P^1, 2\omega_{FS})$$

$$\downarrow \ \pi([z_0 : z_1]) = 2\frac{|z_1|^2}{\|z\|^2}$$

$$\mathbb{R}$$

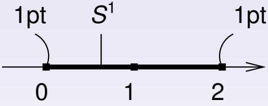

Figure: image of π

Fact

$$\operatorname{ind} D (= \mathrm{H}^0(M; \mathcal{O}_L)) = \# \operatorname{Im} \pi \cap \mathbb{Z}$$

Observation - $\mathbb{C}P^1$ case

$$
\begin{aligned}
b \in \mathrm{Im}\,\pi \cap \mathbb{Z} &\Leftrightarrow \exists \text{parallel section } (\neq 0) \text{ of } (L, \nabla)|_{\pi^{-1}(b)} \\
&\Leftrightarrow \mathrm{H}^0(\pi^{-1}(b); (L, \nabla)|_{\pi^{-1}(b)}) \neq 0 \ (\text{Note: } (L, \nabla)|_{\pi^{-1}(b)} \text{ flat}) \\
&\Leftrightarrow \mathrm{H}^\bullet(\pi^{-1}(b); (L, \nabla)|_{\pi^{-1}(b)}) \neq 0 \ (\because\ \pi^{-1}(b) : \text{torus}) \\
&\Leftrightarrow \ker \left\{ D_b = d_{L|_{\pi^{-1}(b)}} + d^*_{L|_{\pi^{-1}(b)}} : \Gamma(\wedge^\bullet T^* \pi^{-1}(b) \otimes L|_{\pi^{-1}(b)}) \circlearrowright \right\} \neq 0
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\end{aligned}$$

By bundling D_b w. r. t. b, we can obtain the following structure on $M \setminus \pi^{-1}(\mathbb{Z})$.

Structure on $V = M \setminus \pi^{-1}(\mathbb{Z})$

- $\pi|_V : V \to \pi(V)$ *S^1-bundle*
- $D_{fiber} : \Gamma\left((\wedge^\bullet T^*[\pi] \otimes L)|_V\right) \circlearrowright$ *de Rham operator along fibers of* $\pi|_V$*, i.e.*
  - D_{fiber} *contains only derivatives along fibers of* $\pi|_V$
  - $D_{fiber}|_{\pi^{-1}(b)} = d_{L|_{\pi^{-1}(b)}} + d^*_{L|_{\pi^{-1}(b)}}$ $\forall b \in \pi(V)$
- $\ker(D_{fiber}|_{\pi^{-1}(b)}) = 0$ $\forall b \in \pi(V)$

This is a simplest example of an acyclic compatible fibration.

Observation - $\mathbb{C}P^1 \times \mathbb{C}P^1$ case

$$\begin{tikzcd} L = (H,\nabla)^{\otimes 2} \boxtimes (H,\nabla)^{\otimes 2} \arrow[d] \\ M = (\mathbb{C}P^1, 2\omega_{FS}) \times (\mathbb{C}P^1, 2\omega_{FS}) \arrow[d, "\pi \times \pi"] \\ \mathbb{R}^2 \end{tikzcd}$$

$$W = \wedge^{\bullet} T^* M^{0,1} \otimes L$$

$$D = \sqrt{2}(\bar{\partial} \otimes L + \bar{\partial}^* \otimes L) \colon \Gamma(W) \circlearrowleft$$

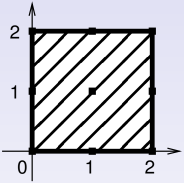

Figure: image of $\pi \times \pi$

Fact

$$\operatorname{ind} D (= \mathrm{H}^0(M; \mathcal{O}_L)) = \# \operatorname{Im}(\pi \times \pi) \cap \mathbb{Z}^2$$

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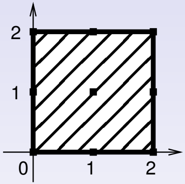

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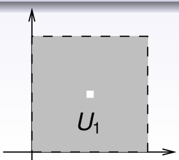
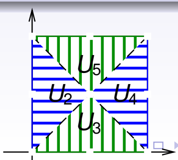
Put $V = M \setminus (\pi \times \pi)^{-1}(\mathbb{Z}^2)$. In this case $\pi|_V$ is no more T^2-bundle. But, locally V has torus bundle structures, i.e.

Observation - $\mathbb{C}P^1 \times \mathbb{C}P^1$ case

Structure on $V_i = (\pi \times \pi)^{-1}(U_i)$ $(i = 1, \dots, 5)$

- $$\begin{tikzcd} V = & V_1 \arrow[d, "\pi_1"] & \cup & V_2 \arrow[d, "\pi_2"] & \cup & V_3 \arrow[d, "\pi_3"] & \cup & V_4 \arrow[d, "\pi_4"] & \cup & V_5 \arrow[d, "\pi_5"] \\ & V_1/S^1_1 \times S^1_2 & & V_2/S^1_2 & & V_3/S^1_1 & & V_4/S^1_2 & & V_5/S^1_1 \end{tikzcd}$$
- $\pi_i^{-1}(\pi_i(V_i \cap V_j)) = \pi_j^{-1}(\pi_j(V_i \cap V_j)) = V_i \cap V_j$
- $$\begin{tikzcd} & V_i \cap V_j \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] \arrow[d, phantom, "\circlearrowright"] & \\ \pi_i(V_i \cap V_j) & & \pi_j(V_i \cap V_j) \arrow[ll, "\exists \pi_{ij}"'] \end{tikzcd}$$
- $D_i : \Gamma\left((\wedge^\bullet T^*[\pi_i] \otimes L|_{V_i})\right) \circlearrowright$ *de Rham operator along fibers of* π_i *with* $\ker(D_i|_{\pi_i^{-1}(b)}) = 0 \ \forall i \ \forall b \in \pi_i(V_i)$





compatible fibration

Definition

A *compatible fibration on* V is the data $\{\pi_\alpha \colon V_\alpha \to U_\alpha\}_{\alpha \in A}$ satisfying

- $V = \cup_\alpha V_\alpha$ is a finite open covering,
- $\forall \alpha\ \pi_\alpha \colon V_\alpha \to U_\alpha$ is a fiber bundle with fiber $(\mathbb{R}/\mathbb{Z})^{k_\alpha}$,
- $\pi_\alpha^{-1}\pi_\alpha(V_\alpha \cap V_\beta) = \pi_\beta^{-1}\pi_\beta(V_\alpha \cap V_\beta) = V_\alpha \cap V_\beta$,
- If $V_\alpha \cap V_\beta \neq \emptyset$ and $\alpha \neq \beta$, then, $\exists \pi_{\alpha\beta} \colon \pi_\beta(V_\alpha \cap V_\beta) \to \pi_\alpha(V_\alpha \cap V_\beta)$ s. t.

$$\begin{tikzcd}
 & V_\alpha \cap V_\beta \arrow[dl, "\pi_\alpha"'] \arrow[dr, "\pi_\beta"] \arrow[d, phantom, "\circlearrowright"] & \\
\pi_\alpha(V_\alpha \cap V_\beta) & & \pi_\beta(V_\alpha \cap V_\beta), \arrow[ll, "\pi_{\alpha\beta}"']
\end{tikzcd}$$

(or $\exists \pi_{\beta\alpha} \colon \pi_\alpha(V_\alpha \cap V_\beta) \to \pi_\beta(V_\alpha \cap V_\beta)$ s. t. the same diagram commutes except that $\pi_{\alpha\beta}$ is replaced by $\pi_{\beta\alpha}$.)

- For a manifold with torus action we can construct a structure of a compatible fibration.

acyclic compatible fibration

Definition

An *acyclic compatible system (of Dirac-type operators along the fibers of* π_α*)* is a data $\{D_\alpha\}_{\alpha \in A}$ satisfying

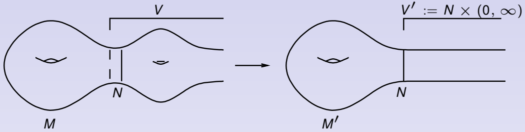
- $D_\alpha \colon \Gamma(W|_{V_\alpha}) \to \Gamma(W|_{V_\alpha})$ is a Dirac-type operator along fibers of π_α,
- $c(\tilde{u}) \circ D_\alpha + D_\alpha \circ c(\tilde{u}) = 0 \ \forall b \in U_\alpha, \forall u \in T_b U_\alpha$,
- $\ker(D_\alpha|_{\pi^{-1}(b)}) = 0 \ \forall b \in U_\alpha$
- $D_\alpha \circ D_\beta + D_\beta \circ D_\alpha$ is a differential operator along the small fibers on $\forall V_\alpha \cap V_\beta \neq \emptyset$,
- $D_\alpha \circ D_\beta + D_\beta \circ D_\alpha$ is non-negative on $\forall V_\alpha \cap V_\beta \neq \emptyset$.

Definition (acyclic compatible fibration)

$$\{(\pi_\alpha \colon V_\alpha \to U_\alpha, D_\alpha)\}_{\alpha \in A}$$

Outline of proof

1. Deform V cylindrically. ($\because$ To get completeness.)



2. For $t \geq 0$ define

$$D_t := D + t \sum_{\alpha \in A} \rho_\alpha D_\alpha \rho_\alpha,$$

where $\{\rho_\alpha^2\}$ is an admissible partition of unity subordinate to $\{V_\alpha\}_{\alpha \in A}$.

Fact & Definition (local index)

1. $\dim \ker D_t \cap L^2 < +\infty$ $(\forall t \gg 0)$.
2. *Moreover,* $\dim \ker D_t^0 \cap L^2 - \dim \ker D_t^1 \cap L^2$ *is independent of* $\forall t \gg 0$.

$$\operatorname{ind}(M, V) := \dim \ker D_t^0 \cap L^2 - \dim \ker D_t^1 \cap L^2 \in \mathbb{Z} \quad (\forall t \gg 0)$$

- $\operatorname{ind}(D_t)$ is localized at $M \setminus V (= \operatorname{supp} \sum_{\alpha \in A} \rho_\alpha D_\alpha \rho_\alpha)$ as $t \to +\infty$.

3. Check $\operatorname{ind}(M, V)$ is independent of a choice of a cut locus.

Point of proof

1. Infinite dimensional analogue of Witten deformation.
2. Use a structure of a fiber bundle with fiber a flat torus instead of a global torus action.
3. Similar to considering an “adiabatic limit"

Examples

Examples

1. $\mathbb{C}P^1$

$$\begin{aligned}\operatorname{ind} D &= \sum_{i=0}^{2} \operatorname{ind}(O_i, O_i \cap V) \\ &= 3 \quad (\because \ \operatorname{ind}(O_i, O_i \cap V) = 1 \ \forall i, \ \text{direct computation})\end{aligned}$$

2. $\mathbb{C}P^1 \times \mathbb{C}P^1$

$$\begin{aligned}\operatorname{ind} D &= \sum_{i=0}^{9} \operatorname{ind}(O_i, O_i \cap V) \\ &= 9 \quad (\because \ \operatorname{ind}(O_i, O_i \cap V) = 1 \ \forall i, \ \text{product formula})\end{aligned}$$

Riemann-Roch number

(L,∇^L) prequantizing line bundle $\left(\frac{\sqrt{-1}}{2\pi}F_\nabla = \omega\right)$

$\downarrow$

(M,ω) closed symplectic manifold

- Fix a compatible almost complex structure J

$$\Rightarrow \begin{cases} 1.\ W = \bigwedge^\bullet T^*M^{0,1} \otimes L \\ 2.\ c\colon TM \to \mathrm{End}\, W \ \ c(u) = \sqrt{2}(\pi^{0,1}(u^*)^\wedge \cdot - \pi^{0,1}(u^*)^\llcorner \cdot) \\ 3.\ \text{Hermitian metric \& Hermitian connection } \nabla \text{ compatible with } c \end{cases}$$

$$\Rightarrow \ D = c \circ \nabla \colon \Gamma(\textstyle\bigwedge^\bullet T^*M^{0,1} \otimes L) \to \Gamma(\textstyle\bigwedge^\bullet T^*M^{0,1} \otimes L)$$

Definition (Riemann-Roch number)

$$RR(M,\omega) = \mathrm{ind}\, D = \dim\ker D^0 - \dim\ker D^1 \in \mathbb{Z}$$

- $RR(M,\omega)$ does not depend on the choice of J.
- If (M,ω) is Kähler and L is holomorphic, then

$$RR(M,\omega) = \sum_i (-1)^i \dim H^i(M,\mathcal{O}_L).$$

Application to Hamiltonian torus actions

$$\begin{tikzcd} & (L,\nabla) \arrow[d] \\ T \curvearrowright & (M,\omega) \end{tikzcd}$$ torus action preserving all data

- $\exists\pi : M \to \mathfrak{t}^*(= \mathrm{Lie}(T)^*)$ moment map defined by

$$\frac{d}{dt}\varphi^*_{e^{t\xi}} s = \nabla_{X_\xi}(\varphi^*_{e^{t\xi}} s) - 2\pi\sqrt{-1}\,\langle \pi, \xi\rangle\, \varphi^*_{e^{t\xi}} s \quad \forall\xi \in \mathfrak{t},\ \forall s \in \Gamma(L).$$

- For $\forall$orbit $\mathcal{O}$, $(L,\nabla)|_{\mathcal{O}}$ is flat.

Theorem

$RR(M,\omega)$ is localized at the inverse image of $\pi(M) \cap \mathfrak{t}^*_{\mathbb{Z}}$.

$$\because\ \ker\left\{d_{L|_{\mathcal{O}}} + d^*_{L|_{\mathcal{O}}} : \Gamma(\wedge^\bullet T^*\mathcal{O} \otimes L|_{\mathcal{O}}) \circlearrowleft\right\} \neq 0 \Leftrightarrow \mathrm{H}^0(\mathcal{O}; (L,\nabla)|_{\mathcal{O}}) \neq 0$$
$$\Rightarrow\ \pi(\mathcal{O}) \in \mathfrak{t}^*_{\mathbb{Z}}.$$

- Theorem does not say anything about local contributions.

Application to Lagrangian fibrations

Lagrangian fibration & BS fiber

$\pi : (M^{2n}, \omega) \to B^n$ Lagrangian fibration

- $\forall$fiber $\cong \mathbb{R}^n/\mathbb{Z}^n$ ($\because$ Arnold-Liouville theorem)
- $(L, \nabla)|_{\text{fiber}}$ is a flat bundle.

Definition (Bohr-Sommerfeld (BS) fiber)

$\pi^{-1}(b)$ ($b \in B$) is said to be *Bohr-Sommerfeld* if $(L, \nabla)|_{\pi^{-1}(b)}$ is trivially flat.

- $\pi^{-1}(b)$ is BS $\Leftrightarrow$ $\exists$non-zero parallel section of $(L, \nabla)|_{\pi^{-1}(b)}$.
- BS fibers appear discretely.

Application to Lagrangian fibrations
$RR(M,\omega) = \#BS$ fibers

"$RR(M,\omega) = \#BS$ fibers" has been observed for

- Lagrangian fibration (Andersen '97)
- Lagrangian fibration with singular fibers
  - moment map of a nonsingular toric variety

$$RR(M,\omega) = \#\pi(M) \cap \mathfrak{t}^*_{\mathbb{Z}} \text{ (Danilov '78)}$$

  - Gelfand-Cetlin completely integrable system on the complex flag manifold (Guillemin-Sternberg '83)
  - Goldman's completely integrable system on the moduli space of flat $SU(2)$-bundles on a Riemann surface (Goldman '86, Jeffrey-Weitsman '92)
  - Pre-symplectic toric manifolds (Karshon-Tolman '93)
  - Torus manifolds (Masuda '99, Hattori-Masuda '03)

  etc.

Question

What mechanism works?

Application to Lagrangian fibrations

Theorem

For a prequantized closed Lagrangian fibration possibly with singular fibers, $RR(M,\omega)$ is localized at singular fibers and BS fibers.

Application to Lagrangian fibrations

Theorem

For a prequantized closed Lagrangian fibration possibly with singular fibers, RR(M, ω) is localized at singular fibers and BS fibers.

Theorem

The contribution of a non-singular BS fiber is one.

Application to Lagrangian fibrations

Theorem

For a prequantized closed Lagrangian fibration possibly with singular fibers, $RR(M, \omega)$ is localized at singular fibers and BS fibers.

Theorem

The contribution of a non-singular BS fiber is one.

Corollary (Andersen '97)

For a prequantized closed Lagrangian fibration without singular fibers,

$$RR(M, \omega) = \#BS \text{ fibers.}$$

Application to Lagrangian fibrations

Theorem

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The contribution of a non-singular BS fiber is one.

Corollary (Andersen '97)

For a prequantized closed Lagrangian fibration without singular fibers,

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Theorem

For a prequantized four-dimensional closed locally toric Lagrangian fibration,

$$RR(M,\omega) = \#\textit{(both singular and nonsingular) BS-fibers}.$$

Comments

1. A systematic method to construct a compatible fibration in general case?

   (Ex. Gelfand-Cetlin system, moduli space of flat $SU(2)$-bundles)
2. Computation of local contributions in general case?
3. $\mathrm{ind}_T(M, V) = ?$