Pluriharmonic maps and submanifolds

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A (simply connected) Kähler manifold M allows for a family of parallel rotations R_θ on its tangent bundle, namely multiplication by the complex scalars $e^{i\theta}$. Given a smooth map $f : M \to S$ into some symmetric space $S = G/K$, one may ask when $df \circ \mathsf{R}_\theta$ is the differential of another map f_θ. When S has semi-definite curvature operator (e.g. when S is compact or dual to a compact symmetric space), this holds if and only if f is pluriharmonic, i.e. the $(1,1)$ part of its hessian vanishes $(\nabla df)^{(1,1)} = \nabla'' d' f = 0$ (without the semi-definitness, the "if" statement is unknown) and $(f_\theta)_{\theta\in[0,2\pi]}$ is called *associated family* of f. Identifying $df \circ \mathsf{R}_\theta$ with df_θ needs an identification of the two tangent spaces $T_{f(x)}S$ and $T_{f_\theta(x)}S$ by applying an element $\Phi_\theta(x) \in G$. This defines a smooth map $\Phi : \mathbb{S}^1 \times M \to G$, the so called *extended solution* (introduced by Uhlenbeck). Further, if a *frame F for f* is given, i.e. a smooth map $F : M \to G$ with $f = \pi \circ F$ for the projection $\pi : G \to G/K$, then $F_\theta = \Phi_\theta F$ is a frame for f_θ, and the family of maps (F_θ) is called an *extended frame* of f. Extended solutions and extended frames are two different descriptions of pluriharmonic maps. We shall discuss the advantages of both notions in various applications.

Extended frames take values in the loop group ΛG. There is some freedom in the choice of F which is fixed by passing to the quotient space $\Lambda G/K$. This can be viewed as *universal twistor space*: Every pluriharmonic map f is the projection of a holomorphic and "superhorizontal" map $\hat{f} : M \to \Lambda G/K$. This infinite dimensional space carries a homogeneous holomorphic structure acted on by the complexified loop group ΛG^c, and this action also preserves the superhorizontal distribution. Applied to $\hat{f}$, these transformations give new pluriharmonic maps which are called *dressing transformations* of f.

An important special case happens when the associated family is constant, $f_\theta = f$ (*isotropic* case). Then $\hat{f}$ takes values in a finite dimensional complex homogeneous subspace of $\Lambda G/K$, a *twistor space*. We will discuss this notion in detail. There are two special cases where all isotropic pluriharmonic maps can be written down explicitly: when S is a complex Grassmannian or a quaternionic symmetric space (Wolf space). For all other compact symmetric spaces S, superhorizontality is a complicated nonholonomic condition, and the general solution seems to be unknown.

The general (non-isotropic) case can be solved by the DPW method (Dorfmeister-Pedit-Wu). However, its application is essentially more difficult than in the surface case ($\dim_{\mathbb{C}} M = 1$) since in higher dimensions it requires solving the *complex curved flat condition*: Finding all closed $\mathfrak{p}^c$-valued one-forms η with $[\eta, \eta] = 0$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g} = \mathrm{Lie}(G)$ corresponding to the symmetric space S and $\mathfrak{p}^c = \mathfrak{p} \otimes \mathbb{C}$. The general solution of this problem seems to be unknown.

Extended solutions take values in the set of based loops $\Omega G = \{\omega : \mathbb{S}^1 \to G;\ \omega(1) = e\}$. One striking application of extended solutions is a reduction of a large class of pluriharmonic maps to the isotropic case: pluriharmonic maps of *finite uniton number*, where Φ is a rational function of $\lambda = e^{-i\theta}$, i.e. its Fourier series in λ is finite. By a theorem of Uhlenbeck and Ohnita-Valli, this holds always if M is compact. Burstall and Guest have shown that Φ can be deformed to an isotropic extended solution Φ^∞ (where the loops $\Phi^\infty(x) : \mathbb{S}^1 \to G$ are group homomorphisms) using the energy flow on ΩG which acts by dressing. In the original paper S was an inner symmetric space, but there is a version for outer symmetric spaces as well. In both cases one obtaines a classification of pluriharmonic maps f by certain isotropic pluriharmonic maps f^∞ in the closure of the dressing orbit of f.

Pluriharmonic maps are useful for submanifold theory in various ways. By a classical theorem of Ruh and Vilms, a surface isometrically immersed in euclidean 3-space has constant mean curvature (CMC) iff its ($\mathbb{S}^2$-valued) Gauss map is harmonic. What are the Kähler submanifolds in euclidean n-space whose (Grassmann-valued) Gauss map is pluriharmonic? They turn out to have parallel *pluri-mean curvature*, "PPMC", i.e. $\nabla \alpha^{(1,1)} = 0$, where the pluri-mean curvature $\alpha^{(1,1)}$ is the (1,1) component of the second fundamental form α. Examples are rare. The only examples known so far are CMC surfaces in $\mathbb{R}^3$ or S^3, pluriminimal submanifold (where $\alpha^{(1,1)} = 0$), and extrinsic hermitian symmetric spaces (where $\nabla \alpha = 0$). It can be shown that further examples must have high codimension.

CMC surfaces in euclidean 3-space yet enjoy another property: They can be computed from their (harmonic) Gauss map, using the associated family. This was discovered by Bonnet and restated differently by Bobenko, using a result of Sym, and it is often called Sym-Bobenko formula. Can one obtain PPMC submanifolds from their (pluriharmonic) Gauss map by a generalized Sym-Bobenko formula? Unfortunately this is not true. However, there is a generalization of this formula where the sphere $\mathbb{S}^2$ is replaced by any hermitian symmetric space of compact type, but instead of PPMC it leads to a new class of Kähler submanifolds which share many properties of CMC surfaces; one might call them "pluri-CMC".