

Weakly reflective orbits and tangentially degenerate orbits of s -representations

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- ① Definitions and fundamental results related to weakly reflective submanifolds
- ② A classification of weakly reflective orbits of s -representations

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Definition of a reflective submanifold

Definition (D. Leung)

Let \widetilde{M} be a complete Riemannian manifold. A connected component M of the fixed point set of an involutive isometry σ of \widetilde{M} is called a **reflective submanifold**.

- A reflective submanifold is a complete totally geodesic submanifold.
- For **all** $x \in M$ and **all** $\xi \in T_x^\perp M$, the reflection σ satisfies

$$\sigma(x) = x, \quad (d\sigma)_x \xi = -\xi, \quad \sigma(M) = M.$$

Definition of a weakly reflective submanifold

Definition

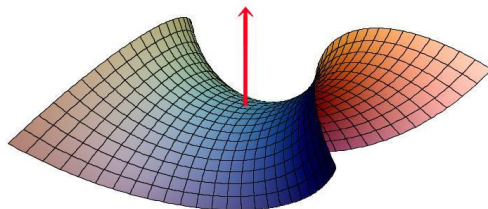
$M \subset \widetilde{M}$: **weakly reflective submanifold (WRS)**

$\stackrel{\text{def}}{\iff}$ for **each** $x \in M$ and **each** $\xi \in T_x^\perp M$,

there exists an isometry σ_ξ of \widetilde{M} which satisfies

$$\sigma_\xi(x) = x, \quad (d\sigma_\xi)_x \xi = -\xi, \quad \sigma_\xi(M) = M.$$

We call σ_ξ a **reflection** of M with respect to ξ .



An example of a weakly reflective submanifold

Example

$$M = S^{n-1}(1) \times S^{n-1}(1) = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n \mid u, v \in S^{n-1}(1)\}$$

is a weakly reflective submanifold in $S^{2n-1}(\sqrt{2})$.

$$x = (e_1, e_1) \in M$$

$$\xi = (e_1, -e_1) \in T_x^\perp M$$

$$\sigma_\xi : S^{2n-1}(\sqrt{2}) \longrightarrow S^{2n-1}(\sqrt{2})$$

$$(u, v) \longmapsto (v, u)$$

Then σ_ξ satisfies

$$\sigma_\xi(x) = x, \quad (d\sigma_\xi)_x \xi = -\xi, \quad \sigma_\xi(M) = M$$

Definition of an austere submanifold

Definition (Harvey-Lawson)

$M \subset \widetilde{M}$: **austere submanifold**

$\stackrel{\text{def}}{\iff}$ for all $\xi \in T_x^\perp M$, the set of eigenvalues of the shape operator A_ξ of M is invariant under the multiplication by -1 , concerning multiplicities.

- An austere submanifolds is a minimal submanifold.
- A minimal surface is an austere submanifold.

Twisted normal bundle (Harvey-Lawson, Borrelli-Gorodski)

$M \subset S^n$: submanifold

$$\begin{aligned}\Phi : N^1 M \times S^1 &\longrightarrow S^{2n+1} \subset \mathbf{R}^{2n+2} \\ (v_x, e^{i\theta}) &\longmapsto (\cos \theta x, \sin \theta v_x)\end{aligned}$$

- A_ξ does not have 0-eigenvalue $\implies \Phi$: Legendrian immersion
- $M \subset S^n$: austere $\implies \Phi$: minimal

Proposition

reflective \subset WRS \subset austere \subset minimal

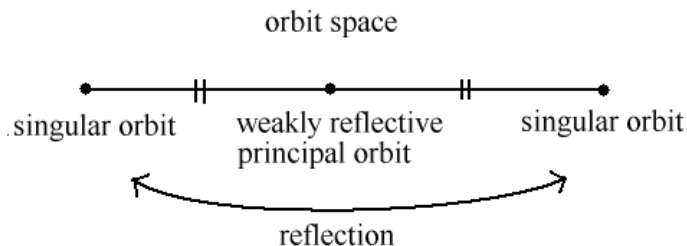
Proposition (Podestà, Ikawa-Tasaki-S.)

Any singular orbit of a cohomogeneity one action on a Riemannian manifold is a weakly reflective submanifold.

Orbits of cohomogeneity one actions

Proposition

Let G be a connected Lie group acting isometrically on a complete, connected Riemannian manifold \tilde{M} . Suppose that the action of G on \tilde{M} is cohomogeneity one with two singular orbits. If there exists a principal orbit which is a weakly reflective submanifold in \tilde{M} , then it has a same distance from two singular orbits and two singular orbits are isometric.



Orbits of s -representations

(G, K) : compact symmetric pair

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$$

The linear isotropy representation $K \curvearrowright \mathfrak{m}$ is called an s -representation. For $H \in S \subset \mathfrak{m}$, the orbit $\text{Ad}(K)H \subset S$ through H is called an R -space.

Related topics

- an orbit of the s -representation of a symmetric space of rank 2 \iff a homogeneous hypersurface in S^n
- G : compact, simple Lie algebra.
 $\text{Ad}(G)H \subset \mathfrak{g} \iff$ a Kähler C -space
- a symmetric R -space \iff a symmetric submanifold in \mathbb{R}^n
- an s -representation \iff a polar representation

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$\mathfrak{a} \subset \mathfrak{m}$: maximal abelian subspace

$$\mathfrak{m}_\lambda = \{X \in \mathfrak{m} \mid [H, [H, X]] = -\langle \lambda, H \rangle^2 X \ (\forall H \in \mathfrak{a})\}$$

$R = \{\lambda \in \mathfrak{a} \mid \mathfrak{m}_\lambda \neq \{0\}\}$: restricted root system

F : fundamental system of R

$$C = \{H \in \mathfrak{a} \mid \langle \alpha, H \rangle > 0 \ (\forall \alpha \in F)\} \text{ : Weyl chamber}$$

$$\text{Ad}(K)\bar{C} = \mathfrak{m} \iff \bar{C} \cap S \text{ : orbit space}$$

For $\Delta \subset F$

$$C^\Delta = \{H \in \bar{C} \mid \langle \alpha, H \rangle > 0 \ (\alpha \in \Delta), \langle \beta, H \rangle = 0 \ (\beta \in F - \Delta)\}$$

$$\bar{C} = \bigcup_{\Delta \subset F} C^\Delta \text{ disjoint union}$$

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Orbits of s -representations

Assume $H \in C^\Delta \subset \bar{C}$

$$\begin{aligned} Z_K^H &= \{k \in K \mid \text{Ad}(K)H = H\} \\ &= \{k \in K \mid \text{Ad}(K)|_{C^\Delta} = \text{id}_{C^\Delta}\} \end{aligned}$$

$$\text{Ad}(K)H \cong K/Z_K^H$$

$$\Delta_1 \subset \Delta_2 \subset F \iff C^{\Delta_1} \subset \bar{C}^{\Delta_2}$$

$$H_1 \in C^{\Delta_1}, H_2 \in C^{\Delta_2} \implies Z_K^{H_1} \supset Z_K^{H_2}$$

Theorem (Hirohashi-Song-Takagi-Tasaki)

For a subset $\Delta \subset F$ there exists unique $H \in C^\Delta \cap S$ such that $\text{Ad}(K)H$ is a minimal submanifold in the hypersphere S .

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Austere orbits of s -representations

An orbit $\text{Ad}(K)H$ of an irreducible s -representation which is an austere submanifold in the hypersphere is one of the following list:

- 1 An orbit through a restricted root
- 2 $R = A_2$; $H = 2e_1 - e_2 - e_3, e_1 + e_2 - 2e_3$
- 3 $R = A_3$; $H = e_1 + e_2 - e_3 - e_4$
- 4 $R = D_n$; $H = e_1$
- 5 $R = D_4$; $H = e_1 + e_2 + e_3 \pm e_4$
- 6 $R = B_2$ with constant multiplicities; $H = e_1 + \frac{e_1 + e_2}{\sqrt{2}}$
(regular orbit)
- 7 $R = G_2$; $H = \alpha_1 + \frac{\alpha_2}{\sqrt{3}}$ (regular orbit)

Weakly reflective orbits of s -representations

An orbit $\text{Ad}(K)H$ of an irreducible s -representation which is a weakly reflective submanifold in the hypersphere is one of the following list:

- ① An orbit through a restricted root
- ② $R = A_2$; $H = 2e_1 - e_2 - e_3, e_1 + e_2 - 2e_3$
- ③ $R = A_3$; $H = e_1 + e_2 - e_3 - e_4$
- ④ $R = D_n$; $H = e_1$
- ⑤ $R = D_4$; $H = e_1 + e_2 + e_3 \pm e_4$

Gauss mapping

$f : M^l \longrightarrow S^n$ immersion

$$\begin{aligned} \text{Gauss map } \gamma : M &\longrightarrow G_{l+1}(\mathbb{R}^{n+1}) \\ x &\longmapsto \mathbb{R}f(x) \oplus T_{f(x)}(f(M)) \end{aligned}$$

Definition

A submanifold $f(M) \subset S^n$ is said to be **tangentially degenerate** if the Gauss map γ is degenerate.

Ferus inequality

Theorem (Ferus)

M^l : compact, connected manifold, $f : M \rightarrow S^n$: immersion

Then , $\text{rank } \gamma < F(l) \implies \text{rank } \gamma = 0$

$F(l) = \min\{k \mid A(k) + k \geq l\}$: Ferus number

$A((2k+1)2^{c+4d}) = 2^c + 8d - 1$: Adams number

$(0 \leq c \leq 3, 0 \leq d)$

Problem (Ishikawa-Kimura-Miyaoka)

- 1 Is the inequality $\text{rank } \gamma < F(l)$ best possible for the implication $\text{rank } \gamma = 0$?
- 2 Classify tangentially degenerate immersions $f : M^l \rightarrow S^n$ with $\text{rank } \gamma = F(l)$.

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Tangentially degenerate orbits

Theorem

An orbit $\text{Ad}(K)H$ of an irreducible \mathfrak{s} -representation which is tangentially degenerate is one of the following list:

- 1 H : a long root
- 2 $R = G_2$; H : a short root

Moreover, if λ is such a root, then

$$\ker(d\gamma)_\lambda = \mathfrak{m}_\lambda.$$

- These orbits are weakly reflective submanifolds.

In general, for the Gauss map γ of a submanifold $M \subset S^n$

$$\begin{aligned}\ker(d\gamma)_x &= \{X \in T_x(M) \mid h(X, Y) = 0, \forall Y \in T_x(M)\} \\ &= \bigcap_{\xi \in T_x^\perp(M)} \ker(A_\xi)\end{aligned}$$

For the Gauss map γ of the orbit $\text{Ad}(K)H$

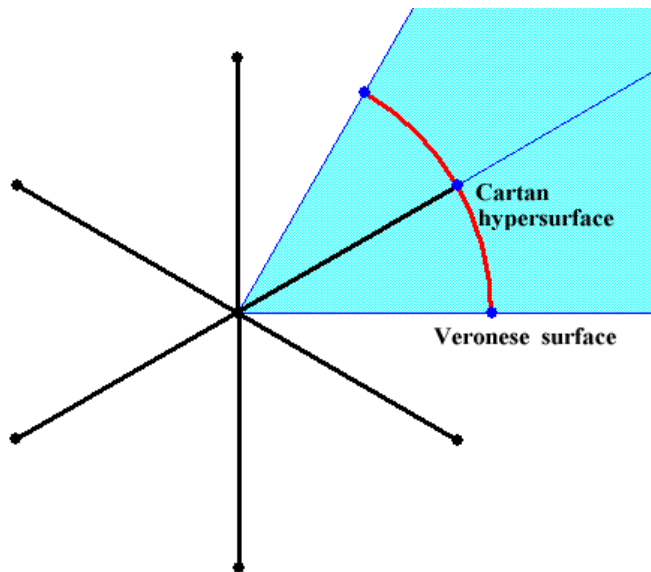
$$\ker(d\gamma)_H = \bigcap_{k \in (Z_K^H)_0} \text{Ad}(k) \sum_{\mu // H} \mathfrak{m}_\mu$$

Proposition

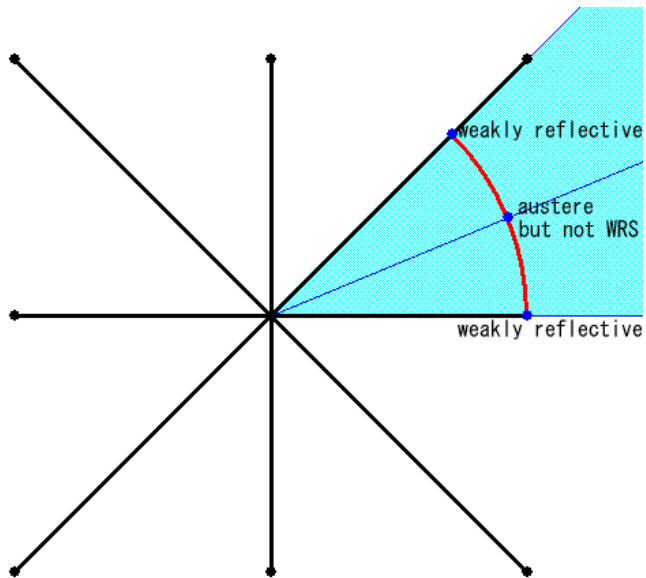
For a restricted root λ , the orbit $\text{Ad}(K)\lambda$ is tangentially degenerate

$$\iff \sum_{\mu // \lambda} \mathfrak{m}_\mu \text{ has non-zero subspace invariant under } \text{ad}(\mathfrak{z}_K^\lambda).$$

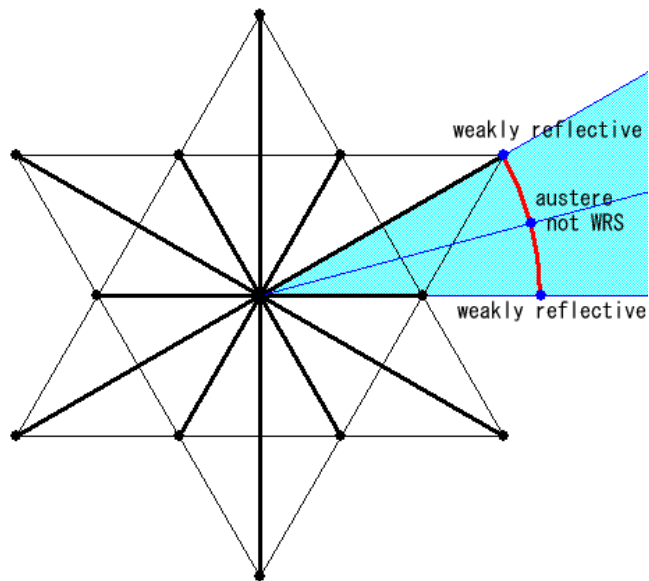
Case of type A_2



Case of type B_2



Case of type G_2



- Ishikawa-Kimura-Miyaoka's problem on the Ferus inequality.
- Classify all weakly reflective submanifolds in Riemannian symmetric spaces.
- Homogeneity of weakly reflective submanifolds.
Construct examples of non-homogeneous weakly reflective submanifolds.
- Study geometry of special Lagrangian submanifolds obtained from austere submanifolds via twisted normal bundle.

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