ON THE TRANSFER PRINCIPLE IN INTEGRAL GEOMETRY

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Abstract. We discuss a generalization of the transfer pinciple in integral geometry. We show that kinematic formulae for integral invariants of degree 2 for hypersurfaces in two point homogeneous spaces can be obtained transferring from the case of real space forms.

1. Introduction

Let M and N be submanifolds in a Riemannian homogeneous space G/K, one fixed and the other moving under the action of $g \in G$. Consider an "*integral invariant*" $I(M \cap gN)$ of the intersection submanifold $M \cap gN$. Then, a formula which express the integral

$$\int_G I(M \cap gN) d\mu_G(g)$$

in terms of some geometric invariants of M and N, where $d\mu_G$ is the invariant measure of G, is called a kinematic formula. One of the most fundamenatal invariants of a submanifold is its volume. A kinemaric formula for volume functional is, especially, called Poincaré formula (see [8] for reference).

Theorem 1.1. *Let M and N be submanifolds of a real space form G/K. If M and N have finite volume then*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p-1})}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})} \mathrm{vol}(M)\mathrm{vol}(N)$$

holds.

Chern [3] and Federer [4] obtained a remarkable kinematic formula as follows:

Theorem 1.2. *Let $I(\mathbb{R}^n)$ denote the isometry group of n dimensional Euclidean space $\mathbb{R}^n$. Assume that $0 \leq 2l \leq p+q-n$. Then there exist constants $c(p,q,n,i,l)$ determined by indicated parameters so that*

$$\int_{I(\mathbb{R}^n)} \mu_{2l}(M \cap gN) d\mu_G(g) = \sum_{i=0}^{l} c(p,q,n,i,l) \mu_{2i}(M) \mu_{2(l-i)}(N)$$

holds for any compact submanifolds M and N in $\mathbb{R}^n$ of dimensions p and q, respectively.

Here the invariants μ_{2i} are from the Weyl tube formula. Definition and some fundamental properties of them will be explained in Section 2.

Later Howard [5] defined integral invariants of submanifolds in Riemannian homogeneous spaces from invariant polynomials on the space of second fundamental

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forms. He showed that kinematic formulae for these invariants can be expressed by invariants of M and N if G is unimodular and acts transitively on the sets of tangent spaces to each of M and N. Moreover he showed the "*transfer principle*" in integral geometry, that guarantees that the same kinematic formulae hold in homogeneous spaces which have same isotropy groups.

The linear isotropy action of a two point homogeneous space is transitive on the sphere in the tangent space at the origin. Therefore kinematic formulae for hypersurfaces in two point homogeneous spaces can be expressed by invariants of two submanifolds. However, it is not obvious how to give explicit forms of such kinematic formulae. In his paper, Howard showed the Poincaré formula in two point homogeneous spaces by transferring from the case of real space forms (Theorem 4.1). In the present paper, we shall study the kimematic formulae for hypersurfaces in two point homogeneous spaces, and show that the kinemaic formulae for integral invariants of degree two can be transferred from the case of real space forms (Theorem 4.3). The discussion here we use is a generalization of the transfer principle.

2. Preliminaries

We shall use this section to recall the general theory of the kinematic formulae in Riemannian homogeneous spaces due to Howard, which is necessary for our discussion. Refer to his paper [5] for details.

Let G be a Lie group and K a compact subgroup of G. We assume that G has a left invariant metric that is also right invariant under K, then G/K is a homogeneous space with an invariant metric. We denote by $T = T_o(G/K)$ the tangent space of G/K at the origin o. Let V be a linear subspace of T. A submanifold M of G/K is said to be of type V if and only if for each $x \in M$ there exists a $g_x \in G$ such that $(g_x)_* V = T_x M$.

For a linear subspace V of T, we define a vector space $\mathrm{II}(V)$ to be

$$\mathrm{II}(V) = \left\{ h \mid h : V \times V \to V^\perp ; \text{symmetric bilinear} \right\},$$

where $V^\perp$ is the normal space of V in T. A second fundamental form of a submanifold of G/K which passes through o and has V as the tangent space at o is an element of $\mathrm{II}(V)$. Let $K(V)$ be the stabilizer of V in K, that is, $K(V) = \{k \in K \,|\, k_* V = V\}$. The group $K(V)$ acts on $\mathrm{II}(V)$ in the following manner:

$$(kh)(u, v) = k_* \left(h(k_*^{-1} u, k_*^{-1} v) \right) \qquad (u, v \in V) \tag{2.1}$$

for $k \in K(V)$ and $h \in \mathrm{II}(V)$. Here we may consider a polynomial $\mathcal{P}$ on the vector space $\mathrm{II}(V)$ which is invariant under $K(V)$, that is, $\mathcal{P}(kh) = \mathcal{P}(h)$ for all $k \in K(V)$ and $h \in \mathrm{II}(V)$. In addition, let M be a submanifold of G/K of type V. For the second fundamental form h_x^M of M at $x \in M$, we define

$$\mathcal{P}(h_x^M) = \mathcal{P}(h_o^{g_x^{-1} M}).$$

Then we can define an integral invariant $I^{\mathcal{P}}(M)$ of M from a polynomial $\mathcal{P}$ by

$$I^{\mathcal{P}}(M) = \int_M \mathcal{P}(h_x^M) d\mu_M(x). \tag{2.2}$$

We also define a vector space $\mathrm{EII}(T)$ to be

$$\mathrm{EII}(T) = \{ h \mid h : T \times T \to T ; \text{symmetric bilinear} \}.$$

Since K also acts on $\mathrm{EII}(T)$ in the same way with (2.1), we can define integral invariants from polynomials on $\mathrm{EII}(T)$ invariant under K in the same manner with (2.2).

With these preliminaries, we can now state the kinematic formulae in Riemannian homogeneous spaces as follows:

Theorem 2.1. ([5] paragraph 4.10) *Let G/K be a Riemannian homogeneous space and assume that G is unimodular. Let V and W be linear subspaces of T with $\dim(V) + \dim(W) \geq \dim(T)$, and $\mathcal{P}$ a homogeneous polynomial of degree l on $\mathrm{EII}(T)$ which is invariant under K such that*

$$\int_K \sigma(V^\perp, k_* W^\perp)^{1-l} d\mu_K(k) < \infty. \tag{2.3}$$

Then there exists a finite set of pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ such that

1. *each $\mathcal{Q}_\alpha$ is a homogeneous polynomial on $\mathrm{II}(V)$ invariant under $K(V)$,*
2. *each $\mathcal{R}_\alpha$ is a homogeneous polynomial on $\mathrm{II}(W)$ invariant under $K(W)$,*
3. $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = l$ *for each α,*
4. *for any compact submanifolds (possibly with boundaries) M of type V and N of type W in G/K the kinematic formula*

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N) \tag{2.4}$$

holds.

Here $\sigma(V, W)$ is the angle between linear subspaces V of dimension p and W of dimension q in an inner product space E. This is defined by

$$\sigma(V, W) = \|v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q\|,$$

where $v_1, \cdots, v_p$ and $w_1, \cdots, w_q$ are orthonormal bases of V and W, respectivery. The inequality (2.3) is a condition for the convergence of the integration. If G/K is a real space form, then the condition (2.3) can be replaced by the manageable inequality $l \leq \dim(M) + \dim(N) - \dim(G/K) + 1$.

In order to explain Theorem 2.1 we need some definitions and lemmas. For $0 < p < n$, $Gr_p(T)$ denotes the Grassmannian manifold of all p dimensional subspaces in T. Then we set

$$\mathrm{II}_p(T) = \{(V, h) \mid V \in Gr_p(T), h \in \mathrm{II}(V)\}.$$

For $(V, h) \in \mathrm{II}_p(T)$ and a subspace W in T with $V + W = T$, we define $G_W(V, h) \in \mathrm{EII}(T)$ by

$$G_W(V, h)(u, v) = P_W^V\left(h(Pu, Pv)\right) \qquad (u, v \in T).$$

Here P_W^V is the projection $T \to (V \cap W)^\perp \cap W$ with kernel V, and P is the orthogonal projection $T \to V \cap W$.

Definition 2.2. Assume that $p + q \geq n$. For $(V, h_1) \in \mathrm{II}_p(T)$, $(W, h_2) \in \mathrm{II}_q(T)$ and a polynomial $\mathcal{P}$ on $\mathrm{EII}(T)$ invariant under K, we define

$$\begin{aligned} &I_K^{\mathcal{P}}(V, h_1, W, h_2) \\ &\quad = \int_K \mathcal{P}\left(G_{k_*^{-1}W}(V, h_1) + G_V(k_*^{-1}W, k^{-1}h_2)\right) \sigma(V^\perp, k_*^{-1}W^\perp) d\mu_K(k) \end{aligned}$$

provided this integral converges.

Lemma 2.3. ([5] paragraph 6.5) *Under the hypothesis of Theorem 2.1 there exists a finite set of pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ such that*

(1) *each* $\mathcal{Q}_\alpha$ *is a homogeneous polynomial on* $\mathrm{II}(V)$ *invariant under* $K(V)$*,*
(2) *each* $\mathcal{R}_\alpha$ *is a homogeneous polynomial on* $\mathrm{II}(W)$ *invariant under* $K(W)$*,*
(3) $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = l$ *for each* α*,*
(4) *for any* $h_1 \in \mathrm{II}(V)$ *and* $h_2 \in \mathrm{II}(W)$

$$I_K^{\mathcal{P}}(V, h_1, W, h_2) = \sum_\alpha \mathcal{Q}_\alpha(h_1)\mathcal{R}_\alpha(h_2).$$

When M and N are submanifolds in G/K of type V and W, we define

$$I_K^{\mathcal{P}}(V, h_x^M, W, h_y^N) = I_K^{\mathcal{P}}(V, h_o^{g_x^{-1}M}, W, h_o^{g_y^{-1}N}).$$

Lemma 2.4. ([5] paragraph 7.2) *Under the hypothesis of Theorem 2.1*

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \int_{M\times N} I_K^{\mathcal{P}}(V, h_x^M, W, h_y^N) d\mu_{M\times N}(x, y)$$

holds for any compact submanifolds M *of type* V *and* N *of type* W *in* G/K*.*

From these two lemmas we conclude Theorem 2.1. Furthermore Lemma 2.4 implies that the integration on G can be reduced to that on K. From this we conclude the following fact.

Transfer principle

Under the hypothesis of Theorem 2.1 let G' be a unimodular Lie group with $\dim G' = \dim G$ and K' be a compact subgroup of G' with isomorphism $\rho : K \to K'$ and $\mathrm{vol}(K) = \mathrm{vol}(K')$. We assume that there is a linear isometry $\psi : T \to T'$ such that

$$\psi \circ k_* = \rho(k)_* \circ \psi \qquad ({}^\forall k \in K).$$

Then ψ induces following isomorphisms of rings:

$$\left\{ \begin{array}{l} \text{polynomials on EII}(T) \\ \text{invariant under } K \end{array} \right\} \cong \left\{ \begin{array}{l} \text{polynomials on EII}(T') \\ \text{invariant under } K' \end{array} \right\}$$
$$\mathcal{P} \longmapsto \mathcal{P}'$$

$$\left\{ \begin{array}{l} \text{polynomials on II}(V) \\ \text{invariant under } K(V) \end{array} \right\} \cong \left\{ \begin{array}{l} \text{polynomials on II}(\psi V) \\ \text{invariant under } K'(\psi V) \end{array} \right\}$$
$$\mathcal{Q}_\alpha,\ \mathcal{R}_\alpha \longmapsto \mathcal{Q}'_\alpha,\ \mathcal{R}'_\alpha$$

Here we denote by $\mathcal{P}', \mathcal{Q}'_\alpha, \mathcal{R}'_\alpha$ the image of $\mathcal{P}, \mathcal{Q}_\alpha, \mathcal{R}_\alpha$ respectively under these isomorphisms. With this notation, if a kinematic formula (2.4) holds in G/K, then a kinematic formula

$$\int_{G'} I^{\mathcal{P}'}(M' \cap gN') d\mu_{G'}(g) = \sum_\alpha I^{\mathcal{Q}'_\alpha}(M') I^{\mathcal{R}'_\alpha}(N')$$

holds for any compact submanifolds M' of type ψV and N' of type ψW in G'/K'.

Now we give some concrete forms of invariant polynomials and kinematic formulae. Take an orthonormal basis $e_1, \dots, e_n$ of T such that $e_1, \dots, e_p$ is a basis of V and $e_{p+1}, \dots, e_n$ is a basis of $V^\perp$. Then the components of $h \in$ II(V) and $H \in$ EII(T) are represented by

$$\begin{aligned} h_{ij}^k &= \langle h(e_i, e_j), e_k \rangle \qquad (1 \le i, j \le p,\ p+1 \le k \le n) \\ H_{ij}^k &= \langle H(e_i, e_j), e_k \rangle \qquad (1 \le i, j, k \le n) \end{aligned}$$

The following polynomials $\mathcal{W}_{2l}$ are homogeneous polynomials on $\mathrm{II}(V)$ of degree $2l$ invariant under $O(V) \times O(V^{\perp})$.

$$\mathcal{W}_{2l}(h) = \sum_{\substack{1 \le i_1, \ldots, i_{2l} \le p \\ p+1 \le k_1, \ldots, k_l \le n}} \det \begin{bmatrix} h^{k_1}_{i_1 i_1} & h^{k_1}_{i_1 i_2} & \cdots & h^{k_1}_{i_1 i_{2l}} \\ h^{k_1}_{i_2 i_1} & h^{k_1}_{i_2 i_2} & \cdots & h^{k_1}_{i_2 i_{2l}} \\ \vdots & \vdots & \ddots & \vdots \\ h^{k_l}_{i_{2l-1} i_1} & h^{k_l}_{i_{2l-1} i_2} & \cdots & h^{k_l}_{i_{2l-1} i_{2l}} \\ h^{k_l}_{i_{2l} i_1} & h^{k_l}_{i_{2l} i_2} & \cdots & h^{k_l}_{i_{2l} i_{2l}} \end{bmatrix}.$$

We define homogeneous polynomials, also denoted by $\mathcal{W}_{2l}$, by

$$\mathcal{W}_{2l}(H) = \sum_{\substack{1 \le i_1, \ldots, i_{2l} \le n \\ 1 \le k_1, \ldots, k_l \le n}} \det \begin{bmatrix} H^{k_1}_{i_1 i_1} & H^{k_1}_{i_1 i_2} & \cdots & H^{k_1}_{i_1 i_{2l}} \\ H^{k_1}_{i_2 i_1} & H^{k_1}_{i_2 i_2} & \cdots & H^{k_1}_{i_2 i_{2l}} \\ \vdots & \vdots & \ddots & \vdots \\ H^{k_l}_{i_{2l-1} i_1} & H^{k_l}_{i_{2l-1} i_2} & \cdots & H^{k_l}_{i_{2l-1} i_{2l}} \\ H^{k_l}_{i_{2l} i_1} & H^{k_l}_{i_{2l} i_2} & \cdots & H^{k_l}_{i_{2l} i_{2l}} \end{bmatrix}$$

on $\mathrm{EII}(T)$ of degree $2l$ invariant under $O(T)$. In the both cases, $\mathcal{W}_0 = 1$ by definitions. A second fundamental form $h \in \mathrm{II}(V)$ can be extended to $H \in \mathrm{EII(T)}$ by

$$H(u, v) = h(Pu, Pv) \qquad (u, v \in T),$$

where $P : T \to V$ is the orthogonal projection. If $H \in \mathrm{EII(T)}$ is the extention of $h \in \mathrm{II}(V)$, then we have

$$\mathcal{W}_{2l}(h) = \mathcal{W}_{2l}(H).$$

Furthermore these polynomials $\mathcal{W}_{2l}$ are characterized as the invariant polynomials which vanish on the (extended) second fundamental forms with relative rank less than $2l$. For a submanifold M of G/K, we introduce the integral invariants $\mu_{2l}(M)$ defined by

$$\mu_{2l}(M) = I^{\mathcal{W}_{2l}}(M).$$

For these integral invariants μ_{2l}, the Chern-Federer kinematic formula (Theorem 1.2) holds. In fact, this formula holds in any real space forms by the transfer principle. The value of the constants $a(p, q, n, i, l)$ were computed by Chern [3] and Nijenhuis [7].

The space of homogeneous polynomials on $\mathrm{II}(V)$ of degree 2 invariant under $O(V) \times O(V^{\perp})$ is spanned by two polynomials

$$\mathcal{Q}_1(h) = \sum_{i,j,k} \left(h^k_{ij}\right)^2, \qquad \mathcal{Q}_2(h) = \sum_k \left(\sum_i h^k_{ii}\right)^2,$$

where $1 \le i, j \le p,\ p+1 \le k \le n$. And if $2 \le p \le n-1$ these two polynomials are independent. Geometrically, $\mathcal{Q}_1(h)$ is the square of the norm of the second fundamental form, and $\mathcal{Q}_2(h)$ is p^2 times the square of the mean curvature. However, it is convenient for us to take the basis

$$\mathcal{W}_2 = \mathcal{Q}_2 - \mathcal{Q}_1, \qquad \mathcal{U}_p = p\mathcal{Q}_1 - \mathcal{Q}_2.$$

For these polynomials we have the following:

Theorem 2.5. *Assume that* $2 \leq p+q-n$. *Then there exist constants* $a(p,q,n)$ *and* $b(p,q,n)$ *so that*

$$\int_G I^{\mathcal{W}_2}(M \cap gN) d\mu_G(g) = a(p,q,n) I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + a(q,p,n)\mathrm{vol}(M) I^{\mathcal{W}_2}(N)$$

$$\int_G I^{\mathcal{U}_{p+q-n}}(M \cap gN) d\mu_G(g) = b(p,q,n) I^{\mathcal{U}_p}(M)\mathrm{vol}(N) + b(q,p,n)\mathrm{vol}(M) I^{\mathcal{U}_q}(N)$$

holds for any compact submanifolds M *and* N *of dimensions* p *and* q *in a real space form* G/K.

The first one is entirely the Chern-Federer formula of degree 2. The second one was suggested by Howard [5]. Finally we gave the explicit forms completely in [6].

The polynomial $\mathcal{U}_p$ is characterized as the invariant polynomial which vanishes at an umbilic point. The integral invariant

$$I^{\mathcal{U}_p^{p/2}}(M) = \int_M \left(\mathcal{U}_p(h_x^M)\right)^{p/2} d\mu_M(x)$$

is an conformal invariant, called the Willmore-Chen functional, of p dimensional submanifold M (see [1], [2], [9]).

3. Two point homogeneous spaces

A connected Riemannian manifold M is called a *two point homogeneous space* if whenever $x_i, y_i \in M$ with distance $d(x_1,y_1) = d(x_2,y_2)$ there exits an isometry $g \in I(M)$ such that $gx_1 = x_2$ and $gy_1 = y_2$. On the other hand, a Riemannian manifold M is said to be *isotropic* at $x \in M$ if $I(M)_x = \{g \in G \mid gx = x\}$ acts transitively on the unit sphere in T_xM, and M is *isotropic* if and only if it is isotropic at every point. It is well-known that these two notions are equivalent. Furthermore, two point homogeneous spaces are completely classified, that is, a Euclidean space $\mathbb{R}^n = I(\mathbb{R}^n)/O(n)$ or an irreducible symmetric space of rank 1. Namely, a sphere $S^n = O(n+1)/O(n)$, a real projective space $\mathbb{R}P^n = O(n+1)/O(1)\times O(n)$, a complex projective space $\mathbb{C}P^n = U(n+1)/U(1)\times U(n)$, a quaternionic projective space $\mathbb{H}P^n = Sp(n+1)/Sp(1)\times Sp(n)$, the Cayley projective plane $\mathbf{Cay}P^2 = F_4/Spin(9)$, and their non-compact duals.

Lemma 3.1. *Let* $M = G/K$ *be a two point homogeneous space. Assume that* G *is the isometry group of* M *listed above. Then there is no homogeneous polynomial on* $\mathrm{II}(V)$ *(resp.* $\mathrm{EII(T)}$*) of odd degree invariant under* $K(V)$ *(resp.* K*).*

Proof. Since $K(V)$ (resp. K) acts on $\mathrm{II}(V)$ (resp. $\mathrm{EII(T)}$) by (2.1), it is enough to find an element $k \in K$ which acts on T as $-\mathrm{id}_T$. It is easy to find such $k \in K$ in the case where $K = O(n), O(1)\times O(n), U(1)\times U(n), Sp(1)\times Sp(n)$. It remaines the case of $K = Spin(9)$. In this case, linear isotropy representation is the spin representation of $Spin(9)$. The spinor group $Spin(n)$ is defined as a subset of the Clifford algebra $Cl(n)$. It is known that a Clifford algebra is isomorphic to a matrix algebra. In this case,

$$Spin(9) \subset Cl(9) \cong M(16,\mathbb{R})$$

where $M(16,\mathbb{R})$ is the algebra of real matrices of 16 by 16. It is obvious that minus the identity matrix $-I \in M(16,\mathbb{R})$ is an element of $Spin(9)$, and this acts on T as $-\mathrm{id}_T$. This completes the proof. □

4. Kinematic formulae in two point homogeneous spaces

Let G/K be a two point homogeneous space of dimension n. Then it is isotropic, that is, K acts transitively on the sphere in T by the linear isotropy representation. Therefore, for $v \in T$ we denote by $K(v)$ the stabilizer of v in K, then $K/K(v)$ is homothetic to the unit sphere S^{n-1}. We note that if we put $W = v^{\perp}$ then $K(v) \subset K(W)$. It is clear that any hypersurfaces in a two point homogeneous space is submanifolds of type V for any hyperplane V in T.

Theorem 4.1. ([5] paragraph 3.12) *Let G/K be a two point homogeneous space of dimension n. Let M be a submanifold of dimension p and N a hypersurface in G/K. If M and N have finite volume then*

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)\mathrm{vol}(S^{p-1})\mathrm{vol}(S^n)}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})} \mathrm{vol}(M)\mathrm{vol}(N)$$

holds.

When G/K is a two point homogeneous space except real space forms, K does not act transitively on $Gr_p(T)$ for $p \geq 2$. Therefore Theorem 4.1 is outside of the general theory, Theorem 2.1. Howard showed this formula in a geometric way like the transfer principle. Here we prove this in a different way using genealized Poincaré formula, that is related to Lemma 2.4.

Let G/K be a Riemannian homogeneous space. For a subspace V in $T_x(G/K)$ and a subspace W in $T_y(G/K)$ we define

$$\sigma_K(V, W) = \int_K \sigma((g_x)_*^{-1} V,\ k_*^{-1} (g_y)_*^{-1} W) d\mu_K(k).$$

Proposition 4.2. ([5] paragraph 3.8) *Let G/K be a Riemannian homogeneous space. Assume that G is unimodular. Then for any compact submanifolds M and N in G/K with* $\dim M + \dim N \geq \dim G/K$

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \int_{M \times N} \sigma_K(T_x^{\perp} M,\ T_y^{\perp} N) d\mu_{M\times N}(x, y) \tag{4.1}$$

holds.

Proof of Theorem 4.1. Let $V \in Gr_p(T)$ and $v \in T$. Set $W = v^{\perp}$. Then

$$\begin{aligned} \sigma_K(V, W) &= \int_K \sigma(V,\ k_* W) d\mu_K(k) \\ &= \mathrm{vol}(K(v)) \int_{K/K(v)} \sigma(V,\ [k]_* W) d\mu_{K/K(v)}([k]) \\ &= \frac{\mathrm{vol}(K)}{\mathrm{vol}(S^{n-1})} \int_{K/K(v)} \sigma(V,\ [k]_* W) d\mu_{S^{n-1}}([k]). \end{aligned}$$

The first equality holds since K is compact. In the second equality the integraltion on K is reduced to that on $K/K(v)$. The last equality means that the invariant measure on $K/K(v)$ is normalized to be the unit sphere S^{n-1}. The last integraltion on the unit sphere is equal for any two point homogeneous spaces. Therefore we have

$$\sigma_K(V, W) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} \sigma_{SO(n)}(V, W).$$

Form Theorem 1.1 we have

$$\sigma_{SO(n)}(V,W) = \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p-1})}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-1})}.$$

This implies the conclusion. □

Poincaré formula is a kinematic formula for the integral invariant of degree 0, that is volume functional. It is a fundamental question how it can be expressed for other integral invariants. From Theorem 2.5 we now show the following formulae.

Theorem 4.3. *Let M and N be real hypersurfaces in a two point homogeneous space G/K. Then the following kinematic formulae hold:*

$$\begin{aligned}
&\int_G I^{\mathcal{W}_2}(M \cap gN) d\mu_G(g) \\
&\quad = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} a(n-1,n-1,n)\big(I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{W}_2}(N)\big), \\
&\int_G I^{\mathcal{U}_{n-2}}(M \cap gN) d\mu_G(g) \\
&\quad = \frac{\mathrm{vol}(K)}{\mathrm{vol}(SO(n))} b(n-1,n-1,n)\big(I^{\mathcal{U}_{n-1}}(M)\mathrm{vol}(N) + \mathrm{vol}(M) I^{\mathcal{U}_{n-1}}(N)\big).
\end{aligned}$$

Proof. Let $\mathcal{P}$ be a polynomial on $\mathrm{EII}(T)$ invariant under the orthogonal group $O(T)$ acting on T. Then, by the definition, for $(V,h_1) \in \mathrm{II}_p(T)$ and $(W,h_2) \in \mathrm{II}_{n-1}(T)$

$$\begin{aligned}
& I_K^{\mathcal{P}}(V,h_1,W,h_2) \\
&\quad = \int_K \mathcal{P}\Big(G_{k_*^{-1}W}(V,h_1) + G_V(k_*^{-1}W, k^{-1}h_2)\Big)\sigma(V^\perp, k_*^{-1}W^\perp) d\mu_K(k) \\
&\quad = \int_K \mathcal{P}\Big(G_{k_*W}(V,h_1) + G_V(k_*W, kh_2)\Big)\sigma(V^\perp, k_*W^\perp) d\mu_K(k) \qquad (4.2)
\end{aligned}$$

The last equality holds since K is a compact Lie group.

We take a second fundamental form $h(r) \in \mathrm{II}(W)$ of a hypersurface which is tangent to W and umbilic at that point with principal curvature r. It is not a problem whether such a hypersurface exists. We are just considering an element of $\mathrm{II}(W)$ formally, i.e., if we take an orthonormal basis of T and regard $\mathrm{II}(W)$ as the space of $(n-1)$ by $(n-1)$ symmetric matrices then $h(r) = rI_{n-1}$, where I_{n-1} is the identity matrix. Since $K(v)$ acts on $\mathrm{II}(W)$ by (2.1), it is clear that $h(r) \in \mathrm{II}(W)$ is fixed by the action of $K(v)$;

$$gh(r) = h(r) \qquad ({}^\forall g \in K(v)). \qquad (4.3)$$

If $[k] = [k'] \in K/K(v)$, then there exists $g \in K(v)$ such that $k' = kg$. Therefore when we apply $h_2 = h(r)$ in (4.2), from (4.3) we have

$$\begin{aligned}
&\mathcal{P}\Big(G_{k'_*W}(V,h_1) + G_V(k'_*W, k'h(r))\Big)\sigma(V^\perp, k'_*W^\perp) \\
&\quad = \mathcal{P}\Big(G_{kg_*W}(V,h_1) + G_V(kg_*W, kgh(r))\Big)\sigma(V^\perp, kg_*W^\perp) \\
&\quad = \mathcal{P}\Big(G_{k_*W}(V,h_1) + G_V(k_*W, kh(r))\Big)\sigma(V^\perp, k_*W^\perp).
\end{aligned}$$

This implies that when we regard K as a principal fiber bundle on $K/K(v)$ with fiber $K(v)$, the integrand in (4.2) is constant on each fiber. Thus the integration on K is reduced to that on $K/K(v)$. Hence we have

$$\begin{aligned} & I_K^{\mathcal{P}}(V,h_1,W,h(r)) \\ &= \operatorname{vol}(K(v))\int_{K/K(v)} \mathcal{P}\Big(G_{[k]_*W}(V,h_1)+G_V([k]_*W,[k]h(r))\Big) \\ &\qquad\qquad \sigma(V^\perp,[k]_*W^\perp)d\mu_{K/K(v)}([k]) \\ &= \frac{\operatorname{vol}(K)}{\operatorname{vol}(S^{n-1})}\int_{K/K(v)} \mathcal{P}\Big(G_{[k]_*W}(V,h_1)+G_V([k]_*W,[k]h(r))\Big) \\ &\qquad\qquad \sigma(V^\perp,[k]_*W^\perp)d\mu_{S^{n-1}}([k]) \\ &= \frac{\operatorname{vol}(K)}{\operatorname{vol}(SO(n))} I_{SO(n)}^{\mathcal{P}}(V,h_1,W,h(r)) \qquad (4.4)\end{aligned}$$

We have the second equality normalizing the invariant measure of $K/K(v)$ to that of unit sphere S^{n-1}. The last equality is obtained by the opposite procedure of reducing the integration on K to that on the sphere.

Now we restrict the condition with $\dim V = \dim W = n-1$, in addition we take an $O(T)$-invariant homogeneous polynomial $\mathcal{P} = \mathcal{W}_{2l}$ on $\mathrm{EII}(T)$. Without loss of generality, we can assume $V = W$, since K acts transitively on $Gr_{n-1}(T)$. In Lemma 3.1 we showed that there is no homogeneous polynomial of odd degree invariant under $K(W)$. Therefore, from Lemma 2.3, there exists a homogeneous polynomial $\mathcal{Q}$ on $\mathrm{II}(W)$ of degree 2 invariant under $K(W)$ so that

$$I_K^{\mathcal{W}_2}(W,h_1,W,h_2) = \mathcal{Q}(h_1)+\mathcal{Q}(h_2).$$

In the case of $K = SO(n)$, this is entirely Theorem 2.5. Thus we have

$$I_{SO(n)}^{\mathcal{W}_2}(W,h_1,W,h_2) = a(n-1,n-1,n)\big(\mathcal{W}_2(h_1)+\mathcal{W}_2(h_2)\big).$$

From (4.4) we have

$$\mathcal{Q}(h_1)+\mathcal{Q}(h(r)) = \frac{\operatorname{vol}(K)}{\operatorname{vol}(SO(n))}a(n-1,n-1,n)\big(\mathcal{W}_2(h_1)+\mathcal{W}_2(h(r))\big).$$

Since $\mathcal{Q}$ is homogeneous polynomial of degree 2 and $h(r) = rI_{n-1}$

$$\mathcal{Q}(h_1)+r^2\mathcal{Q}(h(1)) = \frac{\operatorname{vol}(K)}{\operatorname{vol}(SO(n))}a(n-1,n-1,n)\big(\mathcal{W}_2(h_1)+r^2\mathcal{W}_2(h(1))\big).$$

Here r is arbitrary real number, thus coefficents of polynomials with respect to r agree in each degree. Thus we have

$$\mathcal{Q}(h_1) = \frac{\operatorname{vol}(K)}{\operatorname{vol}(SO(n))}a(n-1,n-1,n)\mathcal{W}_2(h_1).$$

The same discussion holds when we take an invariant polynomial $\mathcal{P} = \mathcal{U}_{n-2}$. Consequently we have the conclusion. □

5. Problems

Finally, we pose some problems related to this paper. In Theorem 4.3 we showed that the kinematic formulae for integral invariants of degree 2 can be obtained transferring from the case of real space forms. Then our interest is in the case of higher degree.

Problem 5.1. Can all kinematic formulae for real hypersurfaces in two point homogeneous spaces for integral invariants defined from $O(T)$-invariant homogeneous polynomials be obtained by transferring from the case of real space forms?

In Theorem 4.1, Howard showed that the Poincaré formula for a real hypersurface N and *any* dimensional submanifold M in a two point homogeneous space. Therefore we pose the following problem naturally:

Problem 5.2. Does Theorem 4.3 hold for a real hypersurface N and any dimensional submanifold M?

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