

Transfer principle in integral geometry

Takashi Sakai

**Department of Mathematics,
Tokyo Metropolitan University**

E-mail : tsakai@comp.metro-u.ac.jp

URL : <http://www.metro-u.ac.jp/~tsakai>

Contents

- Introduction and Examples

Contents

- Introduction and Examples
- Integral invariants

Contents

- Introduction and Examples
- Integral invariants
- Kinematic formula and transfer principle

Contents

- Introduction and Examples
- Integral invariants
- Kinematic formula and transfer principle
- Two point homogeneous spaces

Contents

- Introduction and Examples
- Integral invariants
- Kinematic formula and transfer principle
- Two point homogeneous spaces
- Transferred kinematic formulae in TPHS

Introduction

G/K : Riemannian homogeneous space

M, N : submanifolds of G/K with

$$\dim M + \dim N \geq \dim(G/K)$$

Introduction

G/K : Riemannian homogeneous space

M, N : submanifolds of G/K with

$$\dim M + \dim N \geq \dim(G/K)$$

$$\int_G I(M \cap gN) d\mu_G(g)$$

Introduction

G/K : Riemannian homogeneous space

M, N : submanifolds of G/K with

$$\dim M + \dim N \geq \dim(G/K)$$

$$\int_G I(M \cap gN) d\mu_G(g) = \left\{ \begin{array}{l} \text{geometric invariants} \\ \text{of } M \text{ and } N \end{array} \right\}$$

kinematic formula

Examples

- **Poincaré formula**

$M^p, N^q \subset G/K$: real space form

$$\int_G \text{vol}(M \cap gN) d\mu(g) = C(p, q, n) \text{vol}(M) \text{vol}(N)$$

Examples

- **Poincaré formula**

$M^p, N^q \subset G/K$: real space form

$$\int_G \text{vol}(M \cap gN) d\mu(g) = C(p, q, n) \text{vol}(M) \text{vol}(N)$$

- **Chern-Federer formula**

$M^p, N^q \subset \mathbb{R}^n$

$$0 \leq 2l \leq p + q - n$$

$$\int_G \mu_{2l}(M \cap gN) d\mu_G(g)$$

$$= \sum_{0 \leq k \leq l} C(n, p, q, k, l) \mu_{2k}(M) \mu_{2(l-k)}(N)$$

Integral invariants

G/K : Riemannian homogeneous space

$V \subset T_o(G/K) =: T$

Definition

$M \subset G/K$: submanifold of type V

$\stackrel{\text{def}}{\iff} \exists g_x \in G$ s.t. $(g_x)_*^{-1}(T_x M) = V$ for each
 $x \in M$

Integral invariants

G/K : Riemannian homogeneous space

$V \subset T_o(G/K) =: T$

Definition

$M \subset G/K$: submanifold of type V

$\stackrel{\text{def}}{\iff} \exists g_x \in G$ s.t. $(g_x)_*^{-1}(T_x M) = V$ for each
 $x \in M$

$$K \curvearrowright Gr_p(T)$$

Integral invariants

$$\Pi(V) = \{h : V \times V \rightarrow V^\perp; \text{symmetric bilinear}\}$$

$$K(V) = \{k \in K \mid k_* V = V\}$$

Integral invariants

$$\Pi(V) = \{h : V \times V \rightarrow V^\perp; \text{symmetric bilinear}\}$$

$$K(V) = \{k \in K \mid k_*V = V\}$$

$K(V)$ acts on $\Pi(V)$ by

$$(kh)(u, v) = k_*h(k_*^{-1}u, k_*^{-1}v) \quad (u, v \in V)$$

Integral invariants

$$\Pi(V) = \{h : V \times V \rightarrow V^\perp; \text{symmetric bilinear}\}$$

$$K(V) = \{k \in K \mid k_*V = V\}$$

$K(V)$ acts on $\Pi(V)$ by

$$(kh)(u, v) = k_*h(k_*^{-1}u, k_*^{-1}v) \quad (u, v \in V)$$

\mathcal{P} : polynomial on $\Pi(V)$ invariant under $K(V)$

$M \subset G/K$: submanifold of type V

Integral invariants

$$\Pi(V) = \{h : V \times V \rightarrow V^\perp; \text{symmetric bilinear}\}$$

$$K(V) = \{k \in K \mid k_*V = V\}$$

$K(V)$ acts on $\Pi(V)$ by

$$(kh)(u, v) = k_*h(k_*^{-1}u, k_*^{-1}v) \quad (u, v \in V)$$

\mathcal{P} : polynomial on $\Pi(V)$ invariant under $K(V)$

$M \subset G/K$: submanifold of type V

$$I^{\mathcal{P}}(M) := \int_M \mathcal{P}(h_x^M) d\mu_M$$

Integral invariants

$$\text{EII}(T) = \{H : T \times T \rightarrow T; \text{symmetric bilinear}\}$$

K acts on $\text{EII}(T)$

Integral invariants

$$\text{EII}(T) = \{H : T \times T \rightarrow T; \text{symmetric bilinear}\}$$

K acts on $\text{EII}(T)$

\mathcal{P} : polynomial on $\text{EII}(T)$ invariant under K

$M \subset G/K$: submanifold

$$H_x^M(u, v) := h_x^M(Pu, Pv) \quad (u, v \in T)$$

$$P : T \rightarrow V$$

Integral invariants

$$\text{EII}(T) = \{H : T \times T \rightarrow T; \text{symmetric bilinear}\}$$

K acts on $\text{EII}(T)$

\mathcal{P} : polynomial on $\text{EII}(T)$ invariant under K

$M \subset G/K$: submanifold

$$H_x^M(u, v) := h_x^M(Pu, Pv) \quad (u, v \in T)$$

$$P : T \rightarrow V$$

$$I^{\mathcal{P}}(M) := \int_M \mathcal{P}(H_x^M) d\mu_M$$

Kinematic formula (Howard)

G/K : Riem. homog. space

G : unimodular

$V, W \subset T$, $\dim V + \dim W \geq \dim T$

\mathcal{P} : K -inv. homog. poly. on $\text{EII}(T)$

Then there exist finite pairs $(\mathcal{Q}_\alpha, \mathcal{R}_\alpha)$ which satisfy

(1) \mathcal{Q}_α : $K(V)$ -inv. homog. poly. on $\text{II}(V)$

(2) \mathcal{R}_α : $K(W)$ -inv. homog. poly. on $\text{II}(W)$

(3) $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = \deg \mathcal{P}$ for each α

(4) for any submanifolds M of type V and N of type W in G/K

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_{\alpha} I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N)$$

Generalized kinematic formula

Under the hypothesis of the previous Theorem

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu(g) = \int_{M \times N} I_K^{\mathcal{P}}(V, h_x^M, W, h_y^N) d\mu(x, y)$$

holds for any submanifolds M of type V and N of type W in G/K .

Transfer principle

G'/K' : Riem. homog. spaces, $\dim G' = \dim G$

$\rho : K \rightarrow K'$; isomorphism

$\psi : T_o(G/K) \rightarrow T_{o'}(G'/K')$; linear isometry s.t.

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad (\forall k \in K)$$

Transfer principle

G'/K' : Riem. homog. spaces, $\dim G' = \dim G$

$\rho : K \rightarrow K'$; isomorphism

$\psi : T_o(G/K) \rightarrow T_{o'}(G'/K')$; linear isometry s.t.

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad (\forall k \in K)$$

$$\Rightarrow \left\{ \begin{array}{l} K(V)\text{-inv. poly.} \\ \text{on } \Pi(V) \end{array} \right\} \stackrel{\psi}{\cong} \left\{ \begin{array}{l} K'(\psi V)\text{-inv. poly.} \\ \text{on } \Pi(\psi V) \end{array} \right\}$$

Transfer principle

G'/K' : Riem. homog. spaces, $\dim G' = \dim G$

$\rho : K \rightarrow K'$; isomorphism

$\psi : T_o(G/K) \rightarrow T_{o'}(G'/K')$; linear isometry s.t.

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad (\forall k \in K)$$

$$\Rightarrow \left\{ \begin{array}{l} K(V)\text{-inv. poly.} \\ \text{on } \Pi(V) \end{array} \right\} \stackrel{\psi}{\cong} \left\{ \begin{array}{l} K'(\psi V)\text{-inv. poly.} \\ \text{on } \Pi(\psi V) \end{array} \right\}$$

$$\Rightarrow \int_{G'} I^{\mathcal{P}'}(M' \cap gN') d\mu_{G'}(g) = \sum_{\alpha} I^{\mathcal{Q}'_{\alpha}}(M') I^{\mathcal{R}'_{\alpha}}(N')$$

holds for M' of type ψV and N' of type ψW in G'/K' .

Two point homogeneous spaces

M : two point homogeneous space (TPHS)

$\stackrel{\text{def}}{\iff} x_i, y_i \in M$ with $d(x_1, y_1) = d(x_2, y_2)$
 $\exists g \in \text{Isom}(M)$ s.t. $gx_1 = x_2, gy_1 = y_2$

Two point homogeneous spaces

M : two point homogeneous space (TPHS)

$\stackrel{\text{def}}{\iff} x_i, y_i \in M$ with $d(x_1, y_1) = d(x_2, y_2)$
 $\exists g \in \text{Isom}(M)$ s.t. $gx_1 = x_2, gy_1 = y_2$

M : isotropic

$\stackrel{\text{def}}{\iff} \text{Isom}(M)_x = \{g \in \text{Isom}(M) \mid gx = x\}$
acts transitively on the unit sphere in $T_x M$,
for every $x \in M$

Two point homogeneous spaces

M : two point homogeneous space (TPHS)

$\stackrel{\text{def}}{\iff} x_i, y_i \in M$ with $d(x_1, y_1) = d(x_2, y_2)$
 $\exists g \in \text{Isom}(M)$ s.t. $gx_1 = x_2, gy_1 = y_2$

M : isotropic

$\stackrel{\text{def}}{\iff} \text{Isom}(M)_x = \{g \in \text{Isom}(M) \mid gx = x\}$
acts transitively on the unit sphere in $T_x M$,
for every $x \in M$

M : TPHS $\iff M$: isotropic

$\iff M$: Euclidean space or
rank 1 symmetric space

Poincaré formula in TPHS

G/K : TPHS, $\dim(G/K) = n$

$M^p, N^{n-1} \subset G/K$

$$\int_G \text{vol}(M \cap gN) d\mu_G(g) \\ = \frac{\text{vol}(K) \text{vol}(S^{p-1}) \text{vol}(S^n)}{\text{vol}(S^p) \text{vol}(S^{n-1})} \text{vol}(M) \text{vol}(N)$$

Poincaré formula in TPHS

G/K : TPHS, $\dim(G/K) = n$

$M^p, N^{n-1} \subset G/K$

$$\int_G \text{vol}(M \cap gN) d\mu_G(g) \\ = \frac{\text{vol}(K) \text{vol}(S^{p-1}) \text{vol}(S^n)}{\text{vol}(S^p) \text{vol}(S^{n-1})} \text{vol}(M) \text{vol}(N)$$

Lemma

If G/K is a TPHS, then there is no homogeneous polynomial on $\Pi(V)$ (resp. $\text{EII}(T)$) of odd degree invariant under $K(V)$ (resp. K).

$$\mathcal{W}_{2l}(h) = \sum_{\substack{1 \leq i_1, \dots, i_{2l} \leq p \\ p+1 \leq k_1, \dots, k_l \leq n}} \det \begin{bmatrix} h_{i_1 i_1}^{k_1} & h_{i_1 i_2}^{k_1} & \dots & h_{i_1 i_{2l}}^{k_1} \\ h_{i_2 i_1}^{k_1} & h_{i_2 i_2}^{k_1} & \dots & h_{i_2 i_{2l}}^{k_1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{i_{2l-1} i_1}^{k_l} & h_{i_{2l-1} i_2}^{k_l} & \dots & h_{i_{2l-1} i_{2l}}^{k_l} \\ h_{i_{2l} i_1}^{k_l} & h_{i_{2l} i_2}^{k_l} & \dots & h_{i_{2l} i_{2l}}^{k_l} \end{bmatrix}$$

$O(V) \times O(V^\perp)$ -inv. homog. poly. on $\Pi(V)$
of degree $2l$

$$\mathcal{W}_{2l}(h) = \sum_{\substack{1 \leq i_1, \dots, i_{2l} \leq p \\ p+1 \leq k_1, \dots, k_l \leq n}} \det \begin{bmatrix} h_{i_1 i_1}^{k_1} & h_{i_1 i_2}^{k_1} & \dots & h_{i_1 i_{2l}}^{k_1} \\ h_{i_2 i_1}^{k_1} & h_{i_2 i_2}^{k_1} & \dots & h_{i_2 i_{2l}}^{k_1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{i_{2l-1} i_1}^{k_l} & h_{i_{2l-1} i_2}^{k_l} & \dots & h_{i_{2l-1} i_{2l}}^{k_l} \\ h_{i_{2l} i_1}^{k_l} & h_{i_{2l} i_2}^{k_l} & \dots & h_{i_{2l} i_{2l}}^{k_l} \end{bmatrix}$$

$O(V) \times O(V^\perp)$ -inv. homog. poly. on $\Pi(V)$
of degree $2l$

$$\mu_{2l}(M) := I^{\mathcal{W}_{2l}}(M)$$

- **Hotelling-Weyl tube formula**

$M \subset \mathbb{R}^n$: p -dimensional submanifold

$$\text{vol}(\tau_r M) = \sum_{0 \leq 2l \leq p} C(n, p, l) \mu_{2l}(M) r^{n-p+2l}$$

- **Hotelling-Weyl tube formula**

$M \subset \mathbb{R}^n$: p -dimensional submanifold

$$\text{vol}(\tau_r M) = \sum_{0 \leq 2l \leq p} C(n, p, l) \mu_{2l}(M) r^{n-p+2l}$$

- **Generalized Gauss-Bonnet theorem**

M : compact oriented Riemannian manifold of dimension $2l$

$$\mu_{2l}(M) = C(l) \chi(M)$$

The case of degree 2

The space of homogeneous polynomials of degree 2 on $\Pi(V)$ invariant under $O(V) \times O(V^\perp)$ is spanned by

$$Q_1(h) = \sum_{\substack{1 \leq i, j \leq p \\ p+1 \leq k \leq n}} (h_{ij}^k)^2 = \|h\|^2$$

$$Q_2(h) = \sum_{p+1 \leq k \leq n} \left(\sum_{1 \leq i \leq p} h_{ii}^k \right)^2 = p^2 H^2$$

The case of degree 2

The space of homogeneous polynomials of degree 2 on $\Pi(V)$ invariant under $O(V) \times O(V^\perp)$ is spanned by

$$Q_1(h) = \sum_{\substack{1 \leq i, j \leq p \\ p+1 \leq k \leq n}} (h_{ij}^k)^2 = \|h\|^2$$

$$Q_2(h) = \sum_{p+1 \leq k \leq n} \left(\sum_{1 \leq i \leq p} h_{ii}^k \right)^2 = p^2 H^2$$

$$\mathcal{W}_2(h) = Q_2(h) - Q_1(h)$$

$$\mathcal{U}_p(h) = pQ_1(h) - Q_2(h)$$

Willmore-Chen functional

$$\mathcal{U}_p(h) = \sum_{p+1 \leq k \leq n} \sum_{i < j} (\kappa_i(\xi_k) - \kappa_j(\xi_k))^2$$

Willmore-Chen functional

$$\mathcal{U}_p(h) = \sum_{p+1 \leq k \leq n} \sum_{i < j} (\kappa_i(\xi_k) - \kappa_j(\xi_k))^2$$

$$\langle , \rangle^* = \rho^2 \langle , \rangle \quad \Longrightarrow \quad \kappa_i^*(\xi_k^*) = \rho^{-1} (\kappa_i(\xi_k) - \lambda_k)$$

$\mathcal{U}_p(h) \langle , \rangle$: **conformal invariant**

Willmore-Chen functional

$$\mathcal{U}_p(h) = \sum_{p+1 \leq k \leq n} \sum_{i < j} (\kappa_i(\xi_k) - \kappa_j(\xi_k))^2$$

$$\langle , \rangle^* = \rho^2 \langle , \rangle \quad \Longrightarrow \quad \kappa_i^*(\xi_k^*) = \rho^{-1} (\kappa_i(\xi_k) - \lambda_k)$$

$\mathcal{U}_p(h) \langle , \rangle$: **conformal invariant**

Willmore-Chen functional

$$I\mathcal{U}_p^{p/2}(M) := \int_M (\mathcal{U}_p(h_x^M))^{p/2} d\mu_M(x)$$

Proposition (Chern-Federer, Howard)

G/K : real space form

$$M^p, N^q \subset G/K, \quad 2 \leq p + q - n$$

$$\int_G I^{\mathcal{W}_2}(M \cap gN) dg = a(p, q, n) I^{\mathcal{W}_2}(M) \text{vol}(N) \\ + a(q, p, n) \text{vol}(M) I^{\mathcal{W}_2}(N)$$

$$\int_G I^{\mathcal{U}_{p+q-n}}(M \cap gN) dg = b(p, q, n) I^{\mathcal{U}_p}(M) \text{vol}(N) \\ + b(q, p, n) \text{vol}(M) I^{\mathcal{U}_q}(N)$$

Theorem (Kang-Suh-S.)

In the previous proposition,

$$a(p, q, n) = \frac{p + q - n - 1}{p - 1} \times \frac{\text{vol}(SO(n + 1))\text{vol}(S^{p+q-n})}{\text{vol}(S^p)\text{vol}(S^q)}$$

$$b(p, q, n) = \frac{(p + q - n + 2)(p + q - n - 1)}{(p + 2)(p - 1)} \times \frac{\text{vol}(SO(n + 1))\text{vol}(S^{p+q-n})}{\text{vol}(S^p)\text{vol}(S^q)}$$

Transferred kinematic formulae

Let M and N be real hypersurfaces in a two point homogeneous space G/K . Then the following kinematic formulae hold:

$$\begin{aligned} & \int_G I^{\mathcal{W}_2}(M \cap gN) d\mu_G(g) \\ &= \frac{\text{vol}(K)}{\text{vol}(SO(n))} a(n) (I^{\mathcal{W}_2}(M)\text{vol}(N) + \text{vol}(M)I^{\mathcal{W}_2}(N)) \end{aligned}$$

$$\begin{aligned} & \int_G I^{\mathcal{U}_{n-2}}(M \cap gN) d\mu_G(g) \\ &= \frac{\text{vol}(K)}{\text{vol}(SO(n))} b(n) (I^{\mathcal{U}_{n-1}}(M)\text{vol}(N) + \text{vol}(M)I^{\mathcal{U}_{n-1}}(N)) \end{aligned}$$

Outline of the proof

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \frac{\text{vol}(K)}{\text{vol}(SO(n))} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r))$$

Outline of the proof

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \frac{\text{vol}(K)}{\text{vol}(SO(n))} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r))$$

$\exists Q : K(V)$ -inv. homog. poly. on $\Pi(V)$ of degree 2

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = Q(h_1) + Q(h(r))$$

Outline of the proof

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \frac{\text{vol}(K)}{\text{vol}(SO(n))} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r))$$

$\exists Q : K(V)$ -inv. homog. poly. on $\Pi(V)$ of degree 2

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = Q(h_1) + Q(h(r))$$

$$\begin{aligned} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r)) \\ = a(n-1, n-1, n) (\mathcal{W}_2(h_1) + \mathcal{W}_2(h(r))) \end{aligned}$$

Outline of the proof

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = \frac{\text{vol}(K)}{\text{vol}(SO(n))} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r))$$

$\exists Q : K(V)$ -inv. homog. poly. on $\Pi(V)$ of degree 2

$$I_K^{\mathcal{W}_2}(V, h_1, V, h(r)) = Q(h_1) + Q(h(r))$$

$$\begin{aligned} I_{SO(n)}^{\mathcal{W}_2}(V, h_1, V, h(r)) \\ = a(n-1, n-1, n) (\mathcal{W}_2(h_1) + \mathcal{W}_2(h(r))) \end{aligned}$$

$$\therefore Q = \frac{\text{vol}(K)}{\text{vol}(SO(n))} a(n-1, n-1, n) \mathcal{W}_2$$

Problems

Problem 1

Can all kinematic formulae for hypersurfaces in two point homogeneous spaces for integral invariants defined from $O(T)$ -invariant homogeneous polynomials be obtained by transferring from the case of real space forms?

Problems

Problem 1

Can all kinematic formulae for hypersurfaces in two point homogeneous spaces for integral invariants defined from $O(T)$ -invariant homogeneous polynomials be obtained by transferring from the case of real space forms?

Problem 2

Does Theorem hold for a real hypersurface N and any dimensional submanifold M ?