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Kinematic formulas in Riemannian homogeneous spaces

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G/K : Riemannian homogeneous space

M, N : submanifolds of G/K with

$$\dim M + \dim N \geq \dim(G/K)$$

kinematic formula

$$\int_G I(M \cap gN) d\mu_G(g) = \left\{ \begin{array}{r} \text{geometric invariants} \\ \text{of } M \text{ and } N \end{array} \right\}$$

Poincaré's formula

For any curves c_1 and c_2 in $\mathbb{R}^2$,

$$\int_{I(\mathbb{R}^2)} \#(c_1 \cap gc_2) d\mu(g) = 4 \times \text{length}(c_1) \times \text{length}(c_2)$$

Generalized Poincaré formula (Howard)

G/K : Riemannian homogeneous space

$$\dim(G/K) = n$$

G : unimodular Lie group

M^p, N^q : submanifolds of G/K with $p + q \geq n$

then

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \int_{M \times N} \sigma_K(T_x^\perp M, T_y^\perp N) d\mu(x, y),$$

where

$$\sigma_K(T_x^\perp M, T_y^\perp N) = \int_K \| u_1 \wedge \cdots \wedge u_p \wedge k_*^{-1}(v_1 \wedge \cdots \wedge v_q) \| d\mu_K(k)$$

$u_1, \cdots, u_p$: o.n.b. of $(g_x)_*^{-1}(T_x^\perp M)$

$v_1, \cdots, v_q$: o.n.b. of $(g_y)_*^{-1}(T_y^\perp N)$

V_o : vector subspace of $T_o(G/K)$

$M \subset G/K$: submanifold of <u>type V_o</u>

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists g_x \in G$ s.t. $(g_x)_*^{-1}(T_x M) = V_o$

for each $x \in M$

Corollary

$N \subset G/K$: submanifold of type V_o

$$\Rightarrow \int_G \mathrm{vol}(M \cap gN) d\mu(g)$$
$$= \mathrm{vol}(N) \int_M \sigma_K(T_x^\perp(M),\ V_o^\perp) d\mu(x)$$

In addition $M \subset G/K$: submanifold of type W_o

$$\Rightarrow \int_G \mathrm{vol}(M \cap gN) d\mu(g)$$
$$= \sigma_K(W_o^\perp,\ V_o^\perp)\mathrm{vol}(M)\mathrm{vol}(N)$$

Theorem (Santaló)

$M^p, N^q \subset \mathbb{C}P^n$: complex submanifolds

$$\int_{U(n+1)} \mathrm{vol}(M \cap gN) dg$$
$$= \frac{\mathrm{vol}(\mathbb{C}P^{p+q-n})\mathrm{vol}(U(n+1))}{\mathrm{vol}(\mathbb{C}P^p)\mathrm{vol}(\mathbb{C}P^q)}\mathrm{vol}(M)\mathrm{vol}(N)$$

Theorem (Kang-Takahashi-Tasaki-S.)

G/K : almost Hermitian homogeneous space

$$\dim_{\mathbb{C}}(G/K) = n$$

G : unimodular Lie group

Assume that K acts irreducibly on $\bigwedge^p(T_o(G/K)^{1,0})$.

Then, for any almost complex submanifolds M and N in G/K with $\dim_{\mathbb{C}}(M) = n - p$ and $\dim_{\mathbb{C}}(N) = p$,

$$\int_G \#(M \cap gN) d\mu_G(g) = \frac{\mathrm{vol}(K)}{\binom{n}{p}} \mathrm{vol}(M)\mathrm{vol}(N)$$

holds.

Theorem (S.)

G/K : irreducible Hermitian symmetric space

$$\dim_{\mathbb{C}}(G/K) = n$$

Assume that K acts irreducibly on $\wedge^p(T_o(G/K)^{1,0})$

Then, for any complex submanifolds M and N in G/K with $\dim_{\mathbb{C}}(M) = n-p$, $\dim_{\mathbb{C}}(N) = n-q$ and $p+q \leq n$,

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = \frac{(n-p)!(n-q)!\mathrm{vol}(K)}{n!(n-p-q)!}\mathrm{vol}(M)\mathrm{vol}(N)$$

holds.

Theorem (Chern, Federer)

$M^p, N^q \subset \mathbb{R}^n$: submanifolds

For $0 \le 2l \le p + q - n$,

$$\begin{aligned}&\int_G \mu_{2l}(M \cap gN) d\mu_G(g) \\ &\qquad = \sum_{0 \le k \le l} C(n,p,q,k,l) \mu_{2k}(M) \mu_{2(l-k)}(N)\end{aligned}$$

holds.

Theorem (C. S. Chen)

$M^2, N^2 \subset \mathbb{R}^3$: closed surfaces

$$\begin{aligned}&\int_G \left(\int_{M \cap gN} \kappa^2 ds \right) d\mu_G(g) \\ &\qquad = \pi^3 \mathrm{vol}(N) \int_M \left(\|h^M\|^2 + 2(H^M)^2 \right) d\mu_M \\ &\qquad\quad + \pi^3 \mathrm{vol}(M) \int_N \left(\|h^N\|^2 + 2(H^N)^2 \right) d\mu_N\end{aligned}$$

G/K : Riemannian homogeneous space
$V_o \subset T_o(G/K)$: vector subspace

$$\mathrm{II}(V_o) = \left\{ h \mid h : V_o \times V_o \to V_o^{\perp}; \text{symmetric bilinear} \right\}$$

$$K(V_o) = \{a \in K \mid a_* V_o = V_o\}$$

$K(V_o)$ acts on $\mathrm{II}(V_o)$ by

$$(ah)(u,v) = a_* h(a_*^{-1}u, a_*^{-1}v) \quad (u, v \in V_o)$$

$\mathcal{P}$: polynomial on $\mathrm{II}(V_o)$ invariant under $K(V_o)$
$M \subset G/K$: submanifold of type V_o
For $x \in M$, define

$$\mathcal{P}(h_x^M) := \mathcal{P}(h_o^{g_x^{-1}M}),$$

and

$$I^{\mathcal{P}}(M) := \int_M \mathcal{P}(h_x^M) d\mu_M.$$

$\mathrm{EII}(T_o(G/K))$

$$= \left\{ \begin{array}{r} T_o(G/K) \times T_o(G/K) \to T_o(G/K) \\ ;\text{symmetric bilinear} \end{array} \right\}$$

Theorem (Howard)

G/K : Riemannian homogeneous space

G : unimodular Lie group

$V_o, W_o \subset T_o(G/K)$: vector subspaces with

$$\dim V_o + \dim W_o \geq \dim(G/K)$$

$\mathcal{P}$: homogeneous polynomial on $\mathrm{EII}(T_o(G/K))$ of degree l invariant under K s.t.

$$\int_K \sigma(V_o^{\perp}, k_* W_o^{\perp})^{1-l} d\mu_K(k) < \infty$$

Then there exit $\{\mathcal{Q}_\alpha, \mathcal{R}_\alpha\}_{\alpha \in A}$ which satisfy

(1) $\mathcal{Q}_\alpha$: homogeneous polynomial on $\mathrm{II}(V_o)$ invariant under $K(V_o)$

(2) $\mathcal{R}_\alpha$: homogeneous polynomial on $\mathrm{II}(W_o)$ invariant under $K(W_o)$

(3) $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = l$ for each $\alpha \in A$

(4) for any submanifolds M of type V_o and N of type W_o in G/K

$$\int_G I^{\mathcal{P}}(M \cap gN) d\mu_G(g) = \sum_{\alpha \in A} I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N)$$

holds.

Example

When G/K is a real space form and $\deg \mathcal{P} = 0$, especially $\mathcal{P} \equiv 1$

$$\int_G \mathrm{vol}(M \cap gN) d\mu_G(g) = C \mathrm{vol}(M) \mathrm{vol}(N)$$

Poincaré's formula

Transfer principle

$G/K, G'/K'$: Riemannian homoneneous spaces of same dimensions

Assume that there exist

ρ : $K \to K'$; isomorphism

ψ : $T_o(G/K) \to T_{o'}(G'/K')$; linear isometry s.t.

$$\psi \circ k_* = \rho(k)_* \circ \psi \quad ({}^\forall k \in K)$$

$\Rightarrow \{K(V_o)$-inv. poly. on $II(V_o)\}$

$\cong \{K'(\psi V_o)$-inv. poly. on $II(\psi V_o)\}$

$\Rightarrow$ same type kinematic formulas hold in G/K and G'/K'

The case of real space forms

G/K : real space form

$$\begin{aligned} K &= O(T_o(G/K)) \\ K(V_o) &= O(V_o) \times O(V_o^{\perp}) \end{aligned}$$

There are no homogeneous polynomials of odd degree on $\mathrm{II}(V_o)$ invariant under $K(V_o)$.

$$\begin{aligned} &\mathcal{W}_{2l}(h) \\ &= \sum_{\substack{1 \le i_1, \cdots, i_{2l} \le p \\ 1 \le j_1, \cdots, j_{2l} \le p}} \delta^{i_1, \cdots, i_{2l}}_{j_1, \cdots, j_{2l}} R^{j_1 j_2}_{i_1 i_2}(h) \cdots R^{j_{2l-1} j_{2l}}_{i_{2l-1} i_{2l}}(h) \\ &= 2^l \sum \det \begin{bmatrix} h^{k_1}_{i_1 i_1} & h^{k_1}_{i_1 i_2} & \cdots & h^{k_1}_{i_1 i_{2l}} \\ h^{k_1}_{i_2 i_1} & h^{k_1}_{i_2 i_2} & \cdots & h^{k_1}_{i_2 i_{2l}} \\ \vdots & \vdots & \ddots & \vdots \\ h^{k_l}_{i_{2l-1} i_1} & h^{k_l}_{i_{2l-1} i_2} & \cdots & h^{k_l}_{i_{2l-1} i_{2l}} \\ h^{k_l}_{i_{2l} i_1} & h^{k_l}_{i_{2l} i_2} & \cdots & h^{k_l}_{i_{2l} i_{2l}} \end{bmatrix} \end{aligned}$$

$$\mu_{2l}(M) = I^{\mathcal{W}_{2l}}(M)$$

Generalized Gauss-Bonnet theorem

M : compact oriented Riemannian manifold of dimension $2l$

$$\mu_{2l}(M) = C(l)\chi(M)$$

Hotelling-Weyl tube formula

$M \subset \mathbb{R}^n$: p-dimensional submanifold

$$\mathrm{vol}(\tau_r M) = \sum_{0 \le 2l \le p} C(n,p,l)\mu_{2l}(M) r^{n-p+2l}$$

The case of degree two

The space of homogeneous polynomials of degree two on $II(V_o)$ invariant under $O(V_o)$ is spaned by

$$\begin{aligned}\mathcal{Q}_1(h) &= \sum_{\substack{1\le i,j\le p\\ p+1\le k\le n}} \left(h_{ij}^k\right)^2 = \|h\|^2\\ \mathcal{Q}_2(h) &= \sum_{p+1\le k\le n} \left(\sum_{1\le i\le p} h_{ii}^k\right)^2 = p^2H^2\end{aligned}$$

$$\mathcal{W}_2(h) = 2(\mathcal{Q}_2(h) - \mathcal{Q}_1(h))$$

$$\mathcal{U}_k(h) = k\mathcal{Q}_1(h) - \mathcal{Q}_2(h)$$

Fact (Chern-Federer, Howard)

Assume that $2 \leq p+q-n$. Then there exist constants $a(p,q,n)$ and $b(p,q,n)$ so that for any compact submanifolds M^p and N^q of a real space from G/K the kinematic formulas

$$\begin{aligned}\int_G I^{\mathcal{W}_2}(M \cap gN)dg &= a(p,q,n)I^{\mathcal{W}_2}(M)\mathrm{vol}(N) \\ &\quad +a(q,p,n)\mathrm{vol}(M)I^{\mathcal{W}_2}(N)\end{aligned}$$

$$\begin{aligned}\int_G I^{\mathcal{U}_{p+q-n}}(M \cap gN)dg &= b(p,q,n)I^{\mathcal{U}_p}(M)\mathrm{vol}(N) \\ &\quad +b(q,p,n)\mathrm{vol}(M)I^{\mathcal{U}_q}(N)\end{aligned}$$

holds.

Theorem (Kang-Suh-S.)

In the previous fact,

$$\begin{aligned}a(p,q,n) &= \frac{p+q-n-1}{p-1} \\ &\quad \times \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p+q-n})}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)}, \\ b(p,q,n) &= \frac{(p+q-n+2)(p+q-n-1)}{(p+2)(p-1)} \\ &\quad \times \frac{\mathrm{vol}(SO(n+1))\mathrm{vol}(S^{p+q-n})}{\mathrm{vol}(S^p)\mathrm{vol}(S^q)}.\end{aligned}$$

The case where the intersection is a curve

M^p, N^{n-p+1} : compact submanifolds of a real space form G/K of dimension n.

$$I^{\kappa^2}(M \cap gN) = \int_{M\cap gN} \kappa^2 d\sigma.$$

Proposition

There exist constants $c(p,n)$ and $d(p,n)$ s.t.

$$\begin{aligned}
&\int_G I^{\kappa^2}(M \cap gN)dg \\
&= \Big(c(p,n)I^{\mathcal{W}_2}(M) + d(p,n)I^{\mathcal{U}_p}(M)\Big)\mathrm{vol}(N) \\
&\quad +\mathrm{vol}(M)\begin{pmatrix} c(n-p+1,n)I^{\mathcal{W}_2}(N) \\ +d(n-p+1,n)I^{\mathcal{U}_{n-p+1}}(N)\end{pmatrix}.
\end{aligned}$$

Theorem (Kang-Suh-S.)

In the previous Proposition

$$\begin{aligned}
c(p,n) &= \frac{2\pi}{p-1}\frac{\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-p+1})}, \\
d(p,n) &= \frac{6\pi}{(p+2)(p-1)}\frac{\mathrm{vol}(SO(n+1))}{\mathrm{vol}(S^p)\mathrm{vol}(S^{n-p+1})}.
\end{aligned}$$